Borel completeness of some $\aleph_0$-stable theories

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Abstract

We study $\aleph_0$-stable theories, and prove that if T either has eni-DOP or is eni-deep, then its class of countable models is Borel complete. We introduce the notion of λ-Borel completeness and prove that such theories are λ-Borel complete. Using this, we conclude that an $\aleph_0$-stable theory satisfies $I_{\infty,\aleph_0}(T,\lambda) = 2^\lambda$ for all cardinals λ if and only if T either has eni-DOP or is eni-deep.

1 Introduction and Preliminaries

The main theme of the paper will be to produce many disparate models of an $\aleph_0$-stable theory, assuming some type of non-structure. In all cases, to

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show the complexity of a model M, we concentrate on the regular types $p \in S(M)$ that have *finite dimension in M* i.e., for some (equivalently for every) finite $A \subseteq M$ on which the regular type is based and stationary, we have $\dim(p|A, M)$ finite. That is, there is no infinite, A-independent set of realizations of $p|A$ in M. Clearly, this notion is isomorphism invariant. If f is an isomorphism between M and N, then $p \in S(M)$ has finite dimension in M if and only if $f(p)$ has finite dimension in N. As this is to be our central notion, we recall the following definitions from [11].

Definition 1.1 A stationary, regular type p is *eni (eventually non-isolated)* if there is a finite $A \subseteq \mathrm{dom}(p)$ on which p is based and stationary, yet $p|A$ is not isolated. We say p is ENI if it is both eni and strongly regular.

It is easily checked, see e.g., [11], that a stationary regular type $p \in S(A)$ is eni if and only if it has finite dimension in some model containing A. Recall that for an $\aleph_0$-stable theory, every regular type is non-orthogonal to a strongly regular type. In fact, if M is a model and p is a stationary regular type non-orthogonal to M, then there is a strongly regular $q \in S(M)$ non-orthogonal to p. Hence, if $A \subseteq M$ and $p \in S(A)$ is eni, then there is an ENI $q \in S(M)$ non-orthogonal to p. Furthermore, as strongly regular types are RK-minimal, if a strongly regular $p \in S(M)$ has finite dimension in M, then so does every regular $q \in S(M)$ non-orthogonal to p. Because of this, it would be more precise to state that we code non-structure into a model M by way of which *non-orthogonality classes* of regular types over M have finite dimension.

Our first major result, Theorem 4.12, proves that among $\aleph_0$-stable theories, those possessing eni-DOP are Borel complete. The notion of eni-DOP is defined in Section 2, where a number of notions are shown to be equivalent to it. The existence of a *finite approximation* to a DOP witness (see Subsection 4.1) gives a procedure for constructing a model M_G to code any bipartite graph G. In such a coding, the edge set of G corresponds to the types of finite dimension in M_G. However, it is far from obvious how to recover the vertex set of G from M_G. A weak attempt at this is given in Proposition 4.4, where given an isomorphism f between two models M_G and $M_{G'}$, there is a number ℓ (depending largely on $\mathrm{wt}(f(a)/a)$) so that the image of a complete graph of size $m > \ell$ is almost complete. As the number ℓ depends on the isomorphism and cannot be predicted in advance, we obtain our Borel completeness result by first coding an arbitrary tree $\mathcal{T}$ into a graph $G^*_{\mathcal{T}}$ in which

each node $\eta \in \mathcal{T}$ corresponds to a sequence of finite, complete subgraphs of arbitrarily large size. Then, by composing this map with the coding of graphs into models described above, we obtain a λ-Borel embedding of subtrees of $\lambda^{<\omega}$ into models of our theory.

Once Theorem 4.12 has been established, for the remainder of the paper we assume that T is $\aleph_0$-stable with eni-NDOP. In Section 5 we introduce several notions of decompositions of a given model M. To identify the species, we need to introduce more taxonomy of regular types.

Definition 1.2 A *chain* is a sequence $\langle (M_i, a_i) : i < n^* \leq \omega \rangle$, where, for each i, M_i is a model, $a_i \in M_i$, M_{i+1} is atomic over $M_i \cup \{a_{i+1}\}$, $\mathrm{tp}(a_{i+1}/M_i)$ is regular, and $\mathrm{tp}(a_{i+1}/M_i) \perp M_{i-1}$ when $i > 0$.

Definition 1.3 A stationary, regular type p is *eni-active* if there is a chain $\langle (M_i, a_i) : i \leq n^* < \omega \rangle$ in which p is nonorthogonal to $\mathrm{tp}(a_1/M_0)$ and there is some eni type q non-orthogonal to M_{n^*}.

In particular, every eni type is eni-active.

Definition 1.4 A stationary, regular type p is *dull* if it is not eni-active.

It is readily seen that each of the classes of regular types [regular, eni, eni-active, dull] are closed under non-orthogonality and under automorphisms of the monster model $\mathfrak{C}$. Thus, all four classes are potential candidates for a class $\mathbf{P}$ in [10].

In Definition 5.7, decompositions are named [regular, eni, eni-active] according to the species of $\mathrm{tp}(a_\nu/M_{\nu^-})$. With Theorems 5.10, 5.15, 5.18 we measure the extent to which one can recover a model M from a decomposition of it. Some of these results appear or are implicit in [11] and [3], but are included here to contrast the pros and cons of each species of decomposition.

At first blush, it is not entirely clear how one should define the 'eni-depth' of a theory. By analogy with the classical definition of a deep theory, it should indicate that some species decomposition has an infinite branch, but which one? By looking at the characterization offered by Theorem 7.2, the winner is the following notion.

Definition 1.5 A theory T is *eni-deep* if there is an infinite chain $\langle M_i, a_i : i < \omega \rangle$ in which $\mathrm{tp}(a_{i+1}/M_i)$ is eni-active for every i.

Note that an ω-chain witnessing eni-deepness need not have $\mathrm{tp}(a_{i+1}/M_i)$ eni for any i. Rather, all that is required is that each $\mathrm{tp}(a_{i+1}/M_i)$ be part of some chain with an eni type above it.

In Section 6, with Theorem 6.5, we prove that any $\aleph_0$-stable, eni-deep theory is Borel complete. The proof uses a major result from [10] as a black box.

Finally, in Section 7, we collect our results into Theorem 7.2, that characterizes those $\aleph_0$-stable theories that have maximally large families of $L_{\infty,\aleph_0}$-inequivalent models of every cardinality.

For the whole of this paper, all theories are $\aleph_0$-stable.

1.1 Preliminary facts about $\aleph_0$-stable theories

We begin by enumerating several well-known facts about models of $\aleph_0$-stable theories.

Fact 1.6

1. *Over any set A, prime and atomic models (indeed, constructible) models exist and are unique up to isomorphisms over A;*
2. *If M is a model and $p \not\perp M$, then there is a strongly regular $q \in S(M)$ non-orthogonal to p;*
3. *Strongly regular types over models are RK-minimal, i.e., if $M \preceq N$, $q \in S(M)$ is strongly regular, and there is some $a \in N \setminus M$ such that $\mathrm{tp}(a/M) \not\perp q$, then q is realized in N;*
4. *Any pair $M \preceq N$ of models admits a* strongly regular resolution *i.e., a continuous, elementary chain $\langle M_i : i \leq \alpha \rangle$ of elementary substructures of N such that $M_0 = M$, M_{i+1} is prime over $M_i \cup \{a_i\}$, where* $\mathrm{tp}(a_i/M_i)$ *is strongly regular;*
5. *For any complete type $p \in S(M)$ over a model, there is a finite subset $A \subseteq M$ over which p is based and stationary;*
6. *A model is a-saturated (i.e., $\mathbf{F}^a_{\kappa(T)}$-saturated in the notation of [6]) if and only if it is $\aleph_0$-saturated.*

By combining Fact 1.6(2) and (3), we obtain the very useful '3-model Lemma'.

Lemma 1.7 *Suppose $N_0 \preceq N_1 \preceq M$, $p \in S(N_1)$ is realized in N_1, and is non-orthogonal to N_0. Then there is $q \in S(N_0)$ non-orthogonal to p that is realized in M by an element e satisfying $e \underset{N_0}{\perp} N_1$.*

Proof. By Fact 1.6(2), choose a strongly regular $q \in S(N_0)$ non-orthogonal to p. Let q' be the non-forking extension of q to $S(N_1)$. As p is realized in M, it follows from Fact 1.6(3) that q' is realized in M as well. But any e realizing q' satisfies $e \underset{N_0}{\perp} N_1$.

Next, we give a criterion for λ-saturation of a model of an $\aleph_0$-stable theory. For the moment, call a non-algebraic type $p \in S(M)$ λ*-full* if $\dim(p|A, M) \geq \lambda$ for some (every) finite set $A \subseteq M$ on which p is based and stationary. In particular, a regular type $p \in S(M)$ is of finite dimension if and only if p is not $\aleph_0$-full.

Lemma 1.8 *For λ any infinite cardinal, a model $M \models T$ is λ-saturated if and only if every strongly regular $p \in S(M)$ is λ-full.*

Proof. Left to right is clear, so fix an infinite cardinal λ and a model M in which every strongly regular type is λ-full. If M is not λ-saturated, then there is a subset $A \subseteq M$, $|A| < \lambda$, and a type $q \in S(A)$ that is omitted in M. Among all possible choices, choose q of least Morley rank. Let $q' \in S(M)$ denote the unique non-forking extension of q to M, let a be any realization of q', and let $N = M[a]$ be any prime model over $M \cup \{a\}$. By Fact 1.6(2) there is an element $b \in N \setminus M$ such that $p = \mathrm{tp}(b/M)$ is strongly regular. Choose B, $|B| < \lambda$ such that $A \subseteq B$, p is based and stationary over B, and $\mathrm{tp}(a/Bb)$ forks over B. Since p is λ-full, there is $b^* \in M$ realizing $p|B$. Choose any $a^* \in \mathfrak{C}$ realizing $q|B$ with $\mathrm{tp}(a^*/Bb^*)$ forking over B. Now $a^* \notin M$, lest q be realized in M. Thus, $r = \mathrm{tp}(a^*/M)$ is non-algebraic, yet $MR(r) < MR(q)$, hence $r|C$ is realized in M for any $C \supseteq Bb^*$ on which r is based and stationary and $|C| < \lambda$. However, any realization of $r|C$ is a realization of q, contradicting q being omitted in M.

Given two sets A, B, we say that A *has the Tarski-Vaught property in B,* written $A \subseteq_{TV} B$, if $A \subseteq B$ and every $L(A)$-formula $\varphi(x, a)$ that is realized in B is also realized in A.

Lemma 1.9 *1. If $B \subseteq_{TV} B'$, then for every a, if $\mathrm{tp}(a/B)$ is isolated by the $L(B)$-formula $\varphi(x, b)$, then $\mathrm{tp}(a/B')$ is also isolated by $\varphi(x, b)$.*

2. *Suppose that B and C are sets with B containing a model M and $B \underset{M}{\perp} C$. Then $B \subseteq_{TV} B \cup C$. Furthermore, if A is atomic over B, then $AB \underset{M}{\perp} C$ and A is atomic over C.*

Proof. (1) is Lemma XII 1.12(3) of [6], but we prove it here for convenience. Let $\psi(x, b_1, b')$ be any formula over B' with b_1 from B and b' from B'. Let

$$\theta(y, z, w) := \forall x \forall x'([\varphi(x, y) \wedge \varphi(x', y)] \rightarrow (\psi(x, z, w) \leftrightarrow \psi(x, z, w)))$$

It suffices to show that $\theta(b, b_1, b')$ holds. However, if it failed, then since b, b_1 are from B and $B \subseteq_{TV} B'$, we would have $\neg\theta(b, b_1, b_2)$ for some b_2 from B. But this contradicts $\varphi(x, b)$ isolating $\mathrm{tp}(a/B)$.

(2) That $B \subseteq_{TV} BC$ follows from the finite satisfiability of non-forking over models. That $AB \underset{M}{\perp} C$ is a restatement of isolated types being dominated over models, and the atomicity of A over C follows from (1).

Here is an example of a pair of sets with the Tarski-Vaught property. It is proved in Lemma XII 2.3(3) of [6].

Fact 1.10 *Suppose that M_0, M_1, M_2 are models with $M_1 \underset{M_0}{\perp} M_2$, N_0 is a-saturated and independent from $M_1 M_2$ over N_0, N_1 is a-prime over $N_0 M_1$, and N_2 is a-prime over $N_0 M_2$. Then $M_1 M_2 \subseteq_{TV} N_1 N_2$.*

2 eni-DOP and equivalent notions

We begin with a central notion of [10].

Definition 2.1 A stationary, regular type p has a *DOP witness* if there is a quadruple (M_0, M_1, M_2, M_3) of models, where (M_0, M_1, M_2) form an independent triple of a-models, M_3 is a-prime over $M_1 \cup M_2$, $p \not\perp M_3$, but $p \perp M_1$ and $p \perp M_2$.

Visibly, among stationary, regular types, having a DOP witness is invariant under non-orthogonality.

Recall that by Fact 1.6(6), an a-model is simply an $\aleph_0$-saturated model. As in [10], we are free to vary the amount of saturation of the models (M_0, M_1, M_2). Indeed, as we are working in the $\aleph_0$-stable context, we have even more freedom.

Lemma 2.2 *The following are equivalent for a stationary regular type p.*

1. *p has a DOP witness;*
2. *There is a quadruple (M_0, M_1, M_2, M_3) of models such that (M_0, M_1, M_2) form an independent triple, M_3 is prime over $M_1 \cup M_2$, $p \perp M_1$, $p \perp M_2$, but $p \not\perp M_3$;*
3. *Same as (2), but with $\dim(M_1/M_0)$ and $\dim(M_2/M_0)$ both finite;*
4. *Same as (2), but with $\dim(M_1/M_0) = \dim(M_2/M_0) = 1$*
5. *p has a DOP witness (M_0, M_1, M_2, M_3) (of a-models) with $\dim(M_1/M_0) = \dim(M_2/M_0) = 1$.*

Proof. (1) $\Rightarrow$ (2) is immediate.

(2) $\Rightarrow$ (3): Let (M_0, M_1, M_2, M_3) be any witness to (2). As $p \not\perp M_3$, by Fact 1.6(2) there is a regular $p' \in S(M_3)$ non-orthogonal to p, so without loss, we may assume $p \in S(M_3)$. Choose a finite $d \subseteq M_3$ over which p is based and stationary. As M_3 is atomic over $M_1 \cup M_2$, choose finite $b \subseteq M_1$ and $c \subseteq M_2$ such that $\mathrm{tp}(d/M_0bc) \vdash \mathrm{tp}(d/M_1M_2)$. Let $M_1' \preceq M_1$ be prime over M_0b, let $M_2' \preceq M_2$ be prime over M_0c, and let $M_3' \preceq M_3$ be prime over $M_1' \cup M_2'$. Then $(M_0, M_1'M_2', M_3')$ are as required in (3).

(3) $\Rightarrow$ (4): Assume that (3) holds. Among all possible quadruples of models witnessing (3), choose a triple (M_0, M_1, M_2, M_3) with $\dim(M_1/M_0) + \dim(M_2/M_0)$ as small as possible. Clearly, we cannot have either $M_0 = M_1$ or $M_0 = M_2$, so $\dim(M_1/M_0)$ and $\dim(M_2/M_0)$ are each at least one. We argue that the minimum sum occurs when $\dim(M_1/M_0) = \dim(M_2/M_0) = 1$. Assume this were not the case. Without loss, assume that $\dim(M_1/M_0) \geq 2$. Choose an element $e \in M_1 \setminus M_0$ such that $\mathrm{tp}(e/M_0)$ is strongly regular and let $M_1' \preceq M_1$ be prime over $M_0 \cup \{e\}$. Let $M_3' \preceq M_3$ be prime over $M_1' \cup M_2$. There are two cases. On one hand, if $p \not\perp M_3'$, then the quadruple (M_0, M_1', M_2, M_3') contradicts the minimality of our choice. On the other hand, if $p \perp M_3'$, then the quadruple (M_1', M_1, M_3', M_3) also contradicts the minimality of our choice.

(4) $\Rightarrow$ (5): Let (M_0, M_1, M_2, M_3) be any witness to (4). Without loss, by Fact 1.6(2) we may assume that $p \in S(M_3)$. Choose a finite $d \subseteq M_3$ over which p is based and stationary. Choose a formula $\varphi(x, b, c)$ isolating $\mathrm{tp}(d/M_1M_2)$ with b from M_1 and c from M_2.

Next, choose an a-model N_0 satisfying $N_0 \underset{M_0}{\perp} M_3$, and choose a-prime models N_1 and N_2 over $N_0 \cup M_1$ and $N_0 \cup M_2$, respectively. As $M_3 \underset{M_\ell}{\perp} N_\ell$ for both $\ell = 1, 2$, it follows that $p \perp N_1$ and $p \perp N_2$. Also, as $M_1 M_2 \subseteq_{TV} N_1 N_2$ by Fact 1.10, it follows that $\varphi(x, b, c)$ isolates $\mathrm{tp}(d/N_1 N_2)$ by Lemma 1.9(1). Thus, there is an a-prime model N_3 over $N_1 \cup N_2$ that contains d. Thus, (N_0, N_1, N_2, N_3) is a DOP witness for p with $\dim(N_1/N_0) = \dim(N_2/N_0) = 1$.

(5) $\Rightarrow$ (1) is immediate.

Definition 2.3 T has *eni-DOP* if some eni type p has a DOP witness. Similarly, T has *ENI-DOP* (respectively, *eni-active DOP*) if some ENI-type (respectively, eni-active type) has a DOP witness.

It is fortunate, at least for the exposition, that T having any of the three preceding notions are equivalent. Toward comparing these notions with [10], note that among stationary, regular types, the class of eni types is closed under non-orthogonality and automorphisms of the monster model. Thus, it is a suitable choice of '$\mathbf{P}$' there. With this identification, the class of eni-active types is precisely the class $\mathbf{P}^{\mathrm{active}}$.

In fact, this equivalence extends much further. Recall that a stable theory has the *Omitting Types Order Property (OTOP)* if there is a type $p(x, y, z)$ (where x, y, z denote finite tuples of variables) such that for any cardinal κ there is a model M^* and a sequence $\langle (b_\alpha, c_\alpha) : \alpha < \kappa \rangle$ such that for all $\alpha, \beta < \kappa$,

$$M^* \text{ realizes } p(x, b_\alpha, c_\beta) \quad \text{if and only if} \quad \alpha < \beta$$

Theorem 2.4 *The following are equivalent for an $\aleph_0$-stable theory T:*

1. *T has eni-DOP;*
2. *T has ENI-DOP;*
3. *T has eni-active DOP;*
4. *There is an independent triple (M_0, M_1, M_2) of countable, saturated models such that some (equivalently every) prime model over $M_1 \cup M_2$ is not saturated;*

5. *There is an independent triple (N_0, N_1, N_2) of countable saturated models and strongly regular types $p, q \in S(N_0)$ such that N_1 is $\aleph_0$-prime over N_0 and a realization b of p, N_2 is $\aleph_0$-prime over N_0 and a realization c of q, and if N_3 is prime over N_1N_2, then there is a finite d satisfying $\{b, c\} \subseteq d \subseteq N_3$ and an ENI type $r(x, d)$ that is omitted in N_3 and orthogonal to both N_1 and N_2;*

6. *T has OTOP.*

Proof. The implications (2) $\Rightarrow$ (1) and (1) $\Rightarrow$ (3) are immediate, since every ENI type is eni, and eni type is eni-active. (1) $\Rightarrow$ (2) is clear, since having a DOP witness is closed under non-orthogonality, and every eni type is non-orthogonal to an ENI type. Finally, that (1) $\Rightarrow$ (3) follows from Corollary 3.9 of [10]. Thus, Clauses (1), (2) and (3) are equivalent.

(1) $\Rightarrow$ (4): Suppose that (M_0, M_1, M_2, M_3) is a DOP witness for an eni type p with each model countable saturated. Let $N \preceq M_3$ be prime over $M_1 \cup M_2$, and by way of contradiction, assume that N is saturated. Then as N and M_3 are isomorphic over $M_1 \cup M_2$, by replacing p by a conjugate type, we may assume that $p \in S(N)$. We will contradict the saturation of N by finding a finite subset $B^* \subseteq N$ on which p is based and stationary, but $p|B^*$ is omitted in N.

First, since p is eni, choose a finite $B \subseteq N$ on which p is based and stationary, but $p|B$ is not isolated. Choose finite sets $A_1 \subseteq M_1$ and $A_2 \subseteq M_2$ such that taking $B^* = B \cup A_1 \cup A_2$, we have

$$B^* \underset{A_1A_2}{\perp} M_1M_2 \quad \text{and} \quad B^* \underset{B^* \cap M_0}{\perp} M_0$$

A computation similar to the proof of $(c) \Rightarrow (d)$ in Lemma X, 2.2 of [6] shows that $p|B^* \vdash p|B^*M_1M_2$. Since $p|B^*$ is not isolated, cB^* is not atomic over $M_1 \cup M_2$ for any c realizing $p|B^*$ (hence $p|B^*M_1M_2$). Thus $p|B^*$ is omitted in N.

(4) $\Rightarrow$ (5): Let (M_0, M_1, M_2) exemplify (4), and fix a prime model M_3 over $M_1 \cup M_2$. As M_3 is not saturated, by Lemma 1.8 there is an ENI $r \in S(M_3)$ of finite dimension in M_3.

Claim. r is orthogonal to both M_1 and M_2.

Proof. As the cases are symmetric, assume by way of contradiction that $r \not\perp M_1$. By Fact 1.6(2) there is a strongly regular $p \in S(M_1)$ nonorthogonal to r. Choose a finite $A \subseteq M_3$ such that r is based, stationary and

strongly regular over A, and $r|A$ is omitted in M_3. Choose a finite $B \subseteq M_1$ over which p is based, stationary and strongly regular, and let r' and p' be the unique non-forking extensions of $r|A$ and $p|B$ to AB. Since M_1 is saturated, $\dim(p|B, M_1)$ is infinite, hence $\dim(p', M_3)$ is infinite as well. Thus, $\dim(r', M_3)$ is also infinite, contradicting the fact that $r|A$ is omitted in M_3.

Thus, r has a DOP witness by Lemma 2.2(2). But now, Lemma 2.2(5) gives us the configuration we need.

(5) $\Rightarrow$ (1): Given the triple (M_0, M_1, M_2) and the type r in (5), choose an a-prime model M_3 over $M_1 \cup M_2$. Then (M_0, M_2, M_2, M_3) is a DOP witness for the ENI type r.

(5) $\Rightarrow$ (6): Given the data from (5), let $w(x, u, y, z)$ be the type asserting that y and z are M_0-independent solutions of p and q, respectively, $\varphi(u, y, z)$ isolates $\mathrm{tp}(d/M_1M_2)$ and $r(x, u)$. We argue that the type $\exists u w(x, u, y, z)$ witnesses OTOP. To see this, fix any cardinal κ. Choose $\{b_i : i < \kappa\} \cup \{c_j : j < \kappa\}$ to be M_0-independent, where $\mathrm{tp}(b_i/M_0) = p$ and $\mathrm{tp}(c_j/M_0) = q$ for all $i, j \in \kappa$. For each i, j, let $M_1(b_i)$ be prime over $M_0 \cup \{b_i\}$ and $M_2(c_j)$ be prime over $M_0 \cup \{c_j\}$, and let $\overline{M}$ be prime over the union of these models. Now, for each pair (i, j), choose a witness $d_{i,j}$ to $\varphi(u, b_i, c_j)$ from $\overline{M}$ and let $r_{i,j}$ be shorthand for $r(x, d_{i,j})$. It is easily checked that all of the types $r_{i,j}$ are orthogonal.

For each pair (i, j) with $i \leq j$, choose a realization $e_{i,j}$ of $r_{i,j}$, and let M^* be prime over $\overline{M} \cup \{e_{i,j} : i \leq j < \kappa\}$. Then, because of the orthogonality of the $r_{i,j}$, M^* realizes $\exists uw(x, u, b_i, c_j)$ if and only if $i \leq j$.

(6) $\Rightarrow$ (1): This is Corollary 5.19. (There is no circularity.)

3 λ-Borel completeness

Throughout this section, we **fix a cardinal** $\lambda \geq \aleph_0$. We consider only models of size λ, typically those whose universe is the ordinal λ, in a language of size $\kappa \leq \lambda$. *For notational simplicity, we only consider relational languages.* Although it would be of interest to explore this notion in more generality, here we only study classes $\mathbf{K}$ of L-structures that are closed under $\equiv_{\infty,\aleph_0}$ and study the complexity of $\mathbf{K}/\equiv_{\infty,\aleph_0}$.

Definition 3.1 For any (relational) language L with at most λ symbols, let

$L^{\pm} := L \cup \{\neg R : R \in L\}$, and let S^λ_L denote the set of L-structures M with universe λ. Let

$$L(\lambda) := \{R(\overline{\alpha}) : R \in L^{\pm},\ \overline{\alpha} \in {}^{\mathrm{arity(R)}}\lambda\}$$

and endow S^λ_L with the topology formed by letting

$$\mathcal{B} := \{U_{R(\overline{\alpha})} : R(\overline{\alpha}) \in L(\lambda)\}$$

be a subbasis, where $U_{R(\overline{\alpha})} = \{M \in S^\lambda_L : M \models R(\overline{\alpha})\}$.

Definition 3.2 Given a language L of size at most λ, a set $K \subseteq S^\lambda_L$ is λ-*Borel* if, there is a λ-Boolean combination Ψ of $L(\lambda)$-sentences (i.e., a propositional $L_{\lambda^+,\aleph_0}$-sentence of $L(\lambda)$) such that

$$K = \{M \in S^\lambda_L : M \models \Psi\}$$

Given two languages L_1 and L_2, a function $f : S^\lambda_{L_1} \to S^\lambda_{L_2}$ is λ-*Borel* if the inverse image of every (basic) open set is λ-Borel.

That is, $f : S^\lambda_{L_1} \to S^\lambda_{L_2}$ is λ-Borel if and only if for every $R \in L_2$ and $\overline{\beta} \in {}^{\mathrm{arity(R)}}\lambda$, there is a λ-Boolean combination $\Psi_{R(\overline{\beta})}$ of $L_1(\lambda)$-sentences such that for every $M \in S^\lambda_{L_1}$, $f(M) \models R(\overline{\beta})$ if and only if $M \models \Psi_{R(\overline{\beta})}$.

As two countable structures are isomorphic if and only if they are $\equiv_{\infty,\aleph_0}$, a moment's thought tells us that when $\lambda = \aleph_0$, the notions of $\aleph_0$-Borel sets and functions defined above are equivalent to the usual notion of Borel sets and functions.

Definition 3.3 Suppose that L_1, L_2 are relational languages with at most λ symbols, and for $\ell = 1, 2$, K_ℓ is a λ-Borel subset of $S^\lambda_{L_\ell}$ that is invariant under $\equiv_{\infty,\aleph_0}$. We say that $(K_1, \equiv_{\infty,\aleph_0})$ *is* λ-*Borel reducible to* $(K_2, \equiv_{\infty,\aleph_0})$, written

$$(K_1, \equiv_{\infty,\aleph_0}) \leq^B_\lambda (K_2, \equiv_{\infty,\aleph_0})$$

if there is a λ-Borel function $f : S^\lambda_{L_1} \to S^\lambda_{L_2}$ such that $f(K_1) \subseteq K_2$ and, for all $M, N \in K_1$,

$$M \equiv_{\infty,\aleph_0} N \qquad \text{if and only if} \qquad f(M) \equiv_{\infty,\aleph_0} f(N)$$

Definition 3.4 A class K is *λ-Borel complete for $\equiv_{\infty,\aleph_0}$* if $(K, \equiv_{\infty,\aleph_0})$ is a maximum with respect to $\leq^B_\lambda$. We call a theory T λ-Borel complete for $\equiv_{\infty,\aleph_0}$ if $Mod_\lambda(T)$, the class of models of T with universe λ, is λ-Borel complete for $\equiv_{\infty,\aleph_0}$.

To illustrate this notion, we prove a series of Lemmas, culminating in a generalization of Friedman and Stanley's [2] result that subtrees of $\omega^{<\omega}$ are Borel complete. We make heavy use following characterizations of $\equiv_{\infty,\aleph_0}$-equivalence of structures of size λ.

Fact 3.5 *If $|L| \leq \lambda$, the following conditions are equivalent for L-structures M and N that are both of size λ.*

1. *$M \equiv_{\infty,\aleph_0} N$;*
2. *M and N satisfy the same $L_{\lambda^+,\aleph_0}$-sentences;*
3. *If G is a generic filter of the Levy collapsing poset $Lev(\aleph_0, \lambda)$, then in $V[G]$ there is an isomorphism $h : M \to N$ of countable structures.*

For all $\aleph_0 \leq \kappa \leq \lambda$, let L_κ be the language consisting of the binary relation $\trianglelefteq$ and κ unary predicate symbols $P_i(x)$. Let $\kappa\, CT_\lambda$ denote the class of all L_κ-trees with universe $\lambda^{<\omega}$, colored by the predicates P_i.

Lemma 3.6 *For any (relational) language L satisfying $|L| \leq \kappa \leq \lambda$,*

$$(S^\lambda_L, \equiv_{\infty,\aleph_0}) \leq^B_\lambda (\kappa\, CT_\lambda, \equiv_{\infty,\aleph_0})$$

Proof. For each $n \in \omega$, let $\langle \varphi_{n,i}(\overline{x}) : i < \gamma(n) \leq \kappa \rangle$ be a maximal set of pairwise non-equivalent quantifier-free L-formulas with $\lg(\overline{x}) = n$. As well, fix a bijection $\Phi : \omega \times \kappa \to \kappa$.

Now, given any L-structure $M \in S^\lambda_L$, first note that since the universe of M is λ, the finite sequences from M naturally form a tree isomorphic to $\lambda^{<\omega}$ under initial segment.

So $f(M)$ will consist of this tree, with $\trianglelefteq$ interpreted as the initial segment relation. Furthermore, for each $j \in \kappa$, choose $(n, i) \in \omega \times \kappa$ such that $\Phi(n, i) = j$. If $i < \gamma(n)$, then put

$$P^{f(M)}_j := \{\overline{\alpha} \in \lambda^n : M \models \varphi_{n,i}(\overline{\alpha})\}$$

(if $i \geq \gamma(n)$, then for definiteness, say that P_j always fails on $f(M)$).

Choose any $M, N \in S^\lambda_L$. It is apparent from the construction that if $M \equiv_{\infty,\aleph_0} N$, then $f(M) \equiv_{\infty,\aleph_0} f(N)$. The other direction is more interesting. Suppose that $f(M) \equiv_{\infty,\aleph_0} f(N)$. Consider the Levy collapsing forcing, $Lev(\aleph_0, \lambda)$, that, for any generic filter G, $V[G]$ includes a bijection $g : \omega \to \lambda$. We work in $V[G]$. Note that both $f(M)$ and $f(N)$ are $\equiv_{\infty,\aleph_0}$-equivalent, countable structures. Thus, in $V[G]$, fix an L_κ-isomorphism $h : f(M) \to f(N)$. Using h, in ω steps we construct two branches $\eta, \nu \in \lambda^\omega$, where we think of η as a branch through $f(M)$, while ν is a branch through $f(N)$, satisfying the following three conditions:

- For each $n \in \omega$, $h(\eta(n)) = \nu(n)$;
- $\{g(n) : n \in \omega\} \subseteq \mathrm{dom}(\eta)$; and
- $\{g(n) : n \in \omega\} \subseteq \mathrm{dom}(\nu)$.

Let $F = \{(\eta(n), \nu(n)) : n \in \omega\}$. As $\{g(n) : n \in \omega\}$ is all of λ, it follows that $\mathrm{dom}(F) = \lambda$ and $\mathrm{range}(F) = \lambda$. Furthermore, since $h(\eta(n)) = \nu(n)$, it follows that $P_j(\eta(n)) \leftrightarrow P_j(\nu(n))$ for each j. Thus, for each n, the L-quantifier free types of $\langle \eta(i) : i < n\rangle$ and $\langle \nu(i) : i < n\rangle$ are the same. In particular, it follows that F is a bijection from λ to λ that preserves L-quantifier-free types. Thus, $F : M \to N$ is an isomorphism.

Of course, the isomorphism $F \in V[G]$, but it follows easily by absoluteness that $M \equiv_{\infty,\aleph_0} N$ in V.

Definition 3.7 Given any trees T and $\{S_\eta : \eta \in T\}$, we form the tree $T^*(S_\eta : \eta \in T)$ that 'attaches S_η to T at η' as follows:

The universe of $T^*(S_\eta : \eta \in T)$ (which, for simplicity, we write as T^* below) is the disjoint union of

$$T \sqcup \bigsqcup_{\eta \in T} S_\eta \setminus \{\langle\rangle\}$$

and, for $u, v \in T^*$, we say $u \leq_{T^*} v$ if and only if one of the following clauses hold:

- $u, v \in T$ and $u \trianglelefteq v$; or

- for some $\eta \in T$, $u, v \in \mathcal{S}_\eta \setminus \{\langle\rangle\}$ and $u \trianglelefteq_{S_\eta} v$; or
- $u \in T$, $v \in S_\eta \setminus \{\langle\rangle\}$ and $U \trianglelefteq_T \eta$.

Note that in particular, elements from distinct S_η's are incomparable, and that no element of any S_η is 'below" any element of T. It is easily checked that if T and each of the S_η's are subtrees of $\lambda^{<\omega}$, then the attaching tree $T^*(S_\eta : \eta \in T)$ can also be construed as being a subtree of $\lambda^{<\omega}$.

Definition 3.8 A *subtree of* $\lambda^{<\omega}$ is simply a non-empty subset of $\lambda^{<\omega}$ that is closed under initial segments. Given a subtree T of $\lambda^{<\omega}$, an element $\eta \in T$ is *contained in a branch* if there is some $\nu \in \lambda^\omega$ extending η such that $\nu(n) \in T$ for every $n \in \omega$. A subtree T of $\lambda^{<\omega}$ is *special* if, for every $\eta \in T$ that is contained in a branch, η has no immediate successors that are leaves (i.e., every immediate successor of η has a successor in T).

Lemma 3.9 $(\aleph_0\, CT_\lambda, \equiv_{\infty,\aleph_0}) \leq^B_\lambda$ *(Special subtrees of* $\lambda^{<\omega}, \equiv_{\infty,\aleph_0})$

Proof. Fix a bijection $\Phi : \omega \times \omega \to \omega \setminus \{0, 1\}$. Let T_0 be the tree $\lambda^{<\omega}$.

Also, given any subset $V \subseteq \omega$, let S_V be the rooted tree consisting of one copy of the tree $\omega^{\leq m}$ for each $m \in V$. Other than being joined at the root, the copies of $\omega^{\leq m}$ are disjoint.

Now, suppose we are given $M \in \aleph_0\, CT_\lambda$, i.e., the tree $(\lambda^{<\omega}, \trianglelefteq)$, adjoined by countably many unary predicates $P_j(x)$. We construct a special tree $f(M)$ as follows:

First, form the tree $T_0 = \lambda^{<\omega}$. For each $\eta \in T_0$, let

$$V(\eta) := \{\Phi(n, j) : M \models P_j(\eta)\}$$

where $n = \lg(\eta)$. Note that each $V(\eta) \subseteq \omega \setminus \{0, 1\}$. Let $f(M)$ be the tree $T_0(S_{V(\eta)} : \eta \in T_0)$. By the remark above, as each of T, T_0 and each S_V is a subtree of $\lambda^{<\omega}$, $f(M)$ is also a subtree of $\lambda^{<\omega}$. Furthermore, note that T_0 is recognizable in $f(M)$ as being precisely those elements of $f(M)$ that are contained in an infinite branch. Moreover, for every element $\eta \in f(M)$ that is not contained in an infinite branch, there is a uniform bound on the lengths of $\nu \in f(M)$ extending η. Combining this with the fact that $1 \notin V(\eta)$ for any $(\eta) \in T_0$, we conclude that $f(M)$ is special.

It is easily verified by the construction that if $M \equiv_{\infty,\aleph_0} N$, then $f(M) \equiv_{\infty,\aleph_0} f(N)$. Conversely, suppose that $M, N \in \aleph_0\, CT_\lambda$ and that $f(M) \equiv_{\infty,\aleph_0} f(N)$.

Choose any generic filter G for the Levy collapse $Lev(\aleph_0, \lambda)$. Then, in $V[G]$, there is a tree isomorphism $h : f(M) \to f(N)$ as both $f(M)$ and $f(N)$ are countable and back-and-forth equivalent. It suffices to prove that M and N are isomorphic in $V[G]$.

To see this, first note that since 'being part of an infinite branch' is an isomorphism invariant, the restriction of h to T_0 is a tree isomorphism between the T_0 of M and the T_0 of N. To finish, we need only show that for every $\eta \in T_0$ and $j \in \omega$, $M \models P_j(\eta)$ if and only if $N \models P_j(h(\eta))$. To see this, let $n = \lg(\eta)$ and $k = \Phi(n, j)$. Then $M \models P_j(\eta)$ if and only if there is an immediate successor ν of η that is not part of an infinite branch, but has an extension μ of length $n + k$ that is a leaf. As this condition is also preserved by h, we conclude that $h|_{T_0}$ preserves each of the $\aleph_0$ colors as well.

Corollary 3.10 *There are λ pairwise $\equiv_{\infty,\aleph_0}$-inequivalent special subtrees of $\lambda^{<\omega}$.*

Proof. Let $L = \{R\}$ consist of a single, binary relation, and let DG be the class of all directed graphs (i.e., R-structures) with universe λ. It is well known that there are at least λ pairwise $\equiv_{\infty,\aleph_0}$-inequivalent directed graphs. But, by composing the maps given in Lemmas 3.6 and 3.9, we get a λ-Borel embedding of $(DG, \equiv_{\infty,\aleph_0})$ into (Special subtrees of $\lambda^{<\omega}, \equiv_{\infty,\aleph_0}$) preserving $\equiv_{\infty,\aleph_0}$ in both directions.

Theorem 3.11 *For any infinite cardinal λ, (Subtrees of $\lambda^{<\omega}, \equiv_{\infty,\aleph_0}$) is λ-Borel complete.*

Proof. By Lemma 3.6, it suffices to show

$$(\lambda\, CT_\lambda, \equiv_{\infty,\aleph_0}) \leq^B_\lambda (\text{Subtrees of } \lambda^{<\omega}, \equiv_{\infty,\aleph_0})$$

From the Corollary above, fix a set $\{A_i : i \in \lambda\}$ of pairwise $\equiv_{\infty,\aleph_0}$-inequivalent special subtrees of $\lambda^{<\omega}$.

As notation, let $A_{\langle i \rangle}$ denote the tree A_i, and let $A_{\langle\rangle}$ be the two-element tree $\{\langle\rangle, a\}$ satisfying $\langle\rangle \triangleleft a$. For each $u \subseteq \lambda$, let $T_u = \{\langle\rangle, a\} \cup \{\langle i \rangle : i \in u\}$ and let $S_u = T_u(A_{\langle i \rangle} : i \in u)$. Note that for each $u \subseteq \lambda$, S_u has a unique leaf a attached to $\langle\rangle$, and the trees S_u and S_v are isomorphic if and only if $u = v$.

The proof now follows the proof of Lemma 3.9, using the trees S_u to code the color of a node.

More formally, let $T_0 := \lambda^{<\omega}$ and fix an enumeration $\langle P_j(x) : j \in \lambda \rangle$ of the unary predicates. Given any $M \in \lambda\, CT_\lambda$, for each node $\eta \in T_0$, let $V(\eta) := \{j \in \lambda : M \models V_j(\eta)\}$. Let $f(M)$ be the tree $T_0(S_{V(\eta)} : \eta \in T_0)$.

Note that as each of the A_i's were special, T_0 is detectable in $f(M)$ as being the set of all nodes η that are part of an infinite branch **and** have an immediate successor that is a leaf. The proof now follows Lemma 3.9. In particular, given an isomorphism $h : f(M) \to f(N)$ in $V[G]$, the restriction of h to T_0 is an isomorphism of M and N as $\kappa\, CT_\lambda$-structures.

4 The Borel completeness of $\aleph_0$-stable, eni-DOP theories

This section is devoted to the proofs of Theorem 4.12 and Corollary 4.13. As the proof of the former is lengthy, the section is split into four subsections. The first describes two distinct types of eni-DOP witnesses. The second shows how one can encode bipartite graphs into models of T. However, Proposition 4.4, which gives a bit of positive information about the shapes of the bipartite graphs G and H whenever the associated models M_G and M_H are isomorphic, is rather weak. Thus, instead of trying to recover arbitrary bipartite graphs, in the third subsection we describe how to encode subtrees $\mathcal{T} \subseteq \lambda^{<\omega}$ into bipartite graphs $G^{[m]}_{\mathcal{T}}$, where the nodes of $\mathcal{T}$ correspond to complete, bipartite subgraphs of $G^{[m]}_{\mathcal{T}}$. Finally, in the fourth subsection we prove Theorem 4.12, with Corollary 4.13 following easily from it.

4.1 Two types of eni-DOP witnesses

Fix an eni-DOP witness (M_0, M_1, M_2, M_3, r). A *finite approximation* $\mathcal{F}$ is a 5-tuple (a, b, c, d, r_d), where a, b, c, d are finite tuples from (M_0, M_1, M_2, M_3), respectively, $\mathrm{tp}(b/a)$ and $\mathrm{tp}(c/a)$ are each stationary, regular types, each of b, c contain a and $\{b, c\}$ are independent over a, r is based and stationary on d with $r_d \in S(d)$ parallel to r, and $\mathrm{tp}(d/M_1M_2)$ is isolated by a formula over bc. The last condition, coupled with the fact that M_0, M_1, M_2 are each a-models, yields the following *Extendability Condition:*

$$\mathrm{tp}(d/bc) \vdash \mathrm{tp}(d/b^*c^*)$$

for all $a^* \supseteq a$, $b^* \supseteq ba^*$, $c^* \supseteq ca^*$ such that a^* is independent from bc over a and b^* is independent from c^* over a^*. As well, $r_d \perp b$ and $r_d \perp c$ since r is a witness to eni-DOP.

For a fixed choice $\mathcal{F} = (a, b, c, d, r_d)$ of a finite approximation, the $\mathcal{F}$-*candidates over* a consist of all 4-tuples $(b', c', d', r_{d'})$ such that $\operatorname{tp}(a, b, c, d) = \operatorname{tp}(a, b', c', d')$. There is a natural equivalence relation $\sim_{\mathcal{F}}$ on the $\mathcal{F}$-candidates over a defined by

$$(b, c, d, r_d) \sim_{\mathcal{F}} (b', c', d', r_{d'}) \qquad \text{if and only if} \qquad r_d \not\perp r_{d'}$$

Lemma 4.1 *For any eni-DOP witness (M_0, M_1, M_2, M_3, r), for any finite approximation $\mathcal{F}$, and for any pair (b, c, d, r_d), $(b', c', d', r_{d'})$ of equivalent $\mathcal{F}$-candidates over a, every element of the set $\{b, c, b', c'\}$ depends on the other three over a.*

Proof. Everything is symmetric, so assume by way of contradiction that $b \underset{a}{\perp} cb'c'$. First, as $b'c' \underset{c}{\perp} b$, the Extendibility Condition implies that $\operatorname{tp}(d'/b'c') \vdash \operatorname{tp}(d'/b'c'bc)$. In particular, $d' \underset{b'c'}{\perp} bc$, so $b \underset{c}{\perp} b'c'd'$ follows by the symmetry and transitivity of non-forking.

Second, it follows from this and the Extendibility Condition that $\operatorname{tp}(d/bc) \vdash \operatorname{tp}(d/bcb'c'd')$, so $d \underset{bc}{\perp} b'c'd'$. Combining these two facts yields

$$d \underset{c}{\perp} b'c'd'$$

But then, as $r_d \in S(d)$ is orthogonal to c, by e.g., Claim 1.1 of Chapter X of [6], r_d would be orthogonal to $b'c'd'$, which contradicts $r_d \not\perp r_{d'}$.

It follows from the previous Lemma that there are two types of behavior of a finite approximation $\mathcal{F}$. The following definition describes this dichotomy.

Definition 4.2 Fix an eni-DOP witness (M_0, M_1, M_2, M_3, r). A finite approximation $\mathcal{F} = (a, b, c, d, r_d)$ of it is *flexible* if there is an equivalent $\mathcal{F}$-candidate $(b', c', d', r_{d'})$ over a for which some 3-element subset of $\{b, c, b', c'\}$ is independent over a. We say that the eni-DOP witness (M_0, M_1, M_2, M_3, r) is of *flexible type* if it has a flexible finite approximation. A witness is *inflexible* if it is not flexible.

Lemma 4.3 *Suppose that* (a, b, c, d, r_d) *and* $(a', b', c', d', r_{d'})$ *are each finite approximations of an inflexible eni-DOP witness satisfying* $\mathrm{tp}(a) = \mathrm{tp}(a')$ *and* $r_d \not\perp r_{d'}$. *Then there is no finite set* $A \supseteq aa'$ *satisfying* $\mathrm{tp}(bc/A)$ *does not fork over* a*, exactly one element from* $\{b', c'\}$ *is in* A*, and the other element independent from* A *over* a'.

Proof. By way of contradiction suppose that A were such a set. For definiteness, suppose $b' \in A$ and $c' \underset{a'}{\perp} A$. Let $\mathcal{F}$ denote the finite approximation exemplified by (A, bA, cA, dA, r_{dA}). Fix an automorphism $\sigma \in Aut(\mathfrak{C})$ fixing Ac' pointwise such that $bcd \underset{Ac'}{\perp} \sigma(b)\sigma(c)\sigma(d)$. Then $(\sigma(b)A, \sigma(c)A, \sigma(d)A, r_{\sigma(d)A})$ is an $\mathcal{F}$-candidate over A. Moreover, since $r_{dA} \not\perp r_d \not\perp r_{d'} \not\perp r_{\sigma(d)A}$ the transitivity of non-orthogonality of regular types imply that it is equivalent to (bA, cA, dA, r_{dA}). We will obtain a contradiction to the inflexibility of the eni-DOP witness by exhibiting a 3-element subset of $\{b, c, \sigma(b), \sigma(c)\}$ that is independent over A.

To see this, first note that since b and c are independent over A and $\mathrm{tp}(c'/A)$ has weight 1, c' cannot fork with both b and c over A. For definiteness, suppose that b and c' are independent over A. It follows that $\sigma(b)$ is also independent from c' over A. These facts, together with the independence of b and $\sigma(b)$ over Ac', imply that the three element set $\{b, \sigma(b), c'\}$ is independent over A.

We next claim that $\mathrm{tp}(bc/Ac')$ forks over A. If this were not the case, recalling that $b' \in A$, we would have $bc \underset{aa'}{\perp} b'c'$. Then, by two applications of the Extendibility Condition, we would have $bcd \underset{aa'}{\perp} b'c'd'$, which would contradict $r_d \not\perp r_{d'}$.

But now, the results in the previous two paragraphs, together with the fact that $\mathrm{tp}(c/Ab)$ has weight 1, imply that the set $\{b, \sigma(b), c\}$ is independent over A, contradicting the inflexibility of the eni-DOP witness.

4.2 Coding bipartite graphs into models

In this subsection, we take a particular eni-DOP witness and show how we can embed an arbitrary bipartite graph G into a model M_G. This mapping will be Borel, and isomorphic graphs will give rise to isomorphic models, but the converse is less clear. Proposition 4.4 demonstrates that the graphs

G and H must be similar in some weak sense whenever M_G and M_H are isomorphic.

Fix an eni-DOP witness (M_0, M_1, M_2, M_3, r) and a finite approximation $\mathcal{F} = (a, b, c, d, r_d)$ of it, choosing $\mathcal{F}$ to be flexible if the witness is. As notation, let $p = \text{tp}(b/a)$ and $q = \text{tp}(c/a)$.

We begin by describing how to code arbitrary bipartite graphs into models of T. Given a bipartite graph $G = (L_G, R_G, E_G)$, choose sets $\mathcal{B}_G := \{b_g : g \in L_G\}$ and $\mathcal{C}_G := \{c_h : h \in R_G\}$ such that $\mathcal{B}_G \cup \mathcal{C}_G$ is independent over a, $\text{tp}(b_g/a) = p$ for each $b_g \in \mathcal{B}_G$, and $\text{tp}(c_h/a) = q$ for each $c_h \in \mathcal{C}_G$. As well, for each $(g, h) \in L_G \times R_G$, choose an element $d_{g,h}$ such that $\text{tp}(d_{g,h} b_g c_h/a) = \text{tp}(dbc/a)$ and let $r_{g,h} \in S(d_{g,h})$ be conjugate to r_d. Note that $r_{g,h} \perp r_{g',h'}$ unless $(g, h) = (g', h')$. Let $\mathcal{D}_G = \{d_{g,h} : (g, h) \in E_G\}$ and let $\mathcal{R}_G = \{r_{g,h} : (g, h) \in E_G\}$.

Inductively construct models M_G^n of T as follows: M_G^0 is any prime model over $\mathcal{B}_G \cup \mathcal{C}_G \cup \mathcal{D}_G$. Given M_G^n, let $\mathcal{P}_n = \{p \in S(M_G^n) : p \perp \mathcal{R}_G\}$. By the $\aleph_0$-stability of T, $\mathcal{P}_n$ is countable. Let $\mathcal{E}_n = \{e_s : s \in \mathcal{P}_n\}$ be independent over M_G^n with each e_s realizing s, and let M_G^{n+1} be prime over $M_G^n \cup \mathcal{E}_n$. Finally, let $M_G = \bigcup_{n \in \omega} M_G^n$.

It is easily verified that if G has universe λ, then the mapping $G \mapsto M_G$ is λ-Borel. Moreover, it is easy to see that for regular types $r \in S(M_G)$,

$$r \text{ has finite dimension in } M_G \text{ if and only if } r \not\perp r_{g,h} \text{ for some } (g, h) \in E_G$$

Suppose that $f : M_G \to M_H$ were an isomorphism. Then f maps the regular types in $S(M_G)$ of finite dimension onto the regular types in $S(M_H)$ of finite dimension. Thus, by construction of M_G and M_H, this correspondence yields a bijection

$$\pi_f : E_G \to E_H$$

Unfortunately, this identification need not extend to a bipartite graph isomorphism between G and H. Specifically, there might be edges $e_1, e_2 \in E_G$ that share a vertex of G, while the corresponding edges $\pi_f(e_1), \pi_f(e_2) \in E_H$ do not have a common vertex. The bulk of our argument will be to show that images of sufficiently large, complete bipartite subgraphs of G cannot be too wild.

To make this precise, for $X \subseteq E_G$, let $v_G(X)$ denote the smallest subset of the vertices of G with $X \subseteq E_{v_G(X)}$. For ℓ very large, call a graph G *almost*

ℓ*-complete bipartite* if it is $m_1 \times m_2$ bipartite with $0.99\ell \leq m_i \leq \ell$ for $i = 1, 2$ and each vertex has valence at least 0.9ℓ.

The proof of the following Proposition is substantial, and occupies the remainder of this subsection.

Proposition 4.4 *For any bipartite graphs G and H and for any isomorphism $f : M_G \to M_H$, there is a number ℓ^*, depending only on f, such that for all $\ell \geq \ell^*$, if $G_0 \subseteq G$ is any complete $\ell \times \ell$ bipartite subgraph, then $v_H(\pi_f(E_{G_0}))$ contains an almost ℓ-complete bipartite subgraph.*

Proof. Fix bipartite graphs G, H, and an isomorphism $f : M_G \to M_H$. As notation, let $a' = f^{-1}(a)$, let $\mathcal{B}'_H = \{f^{-1}(b) : b \in \mathcal{B}_H\}$, and let $\mathcal{C}'_H = \{f^{-1}(c) : c \in \mathcal{C}_H\}$. Let $X \subseteq \mathcal{B}_G \cup \mathcal{C}_G$ be minimal such that $\mathrm{tp}(a'/a\mathcal{B}_G \cup \mathcal{C}_G)$ does not fork over Xa, and let $X' \subseteq \mathcal{B}'_H \cup \mathcal{C}'_H$ be minimal such that $\mathrm{tp}(a/a'\mathcal{B}'_H\mathcal{C}'_H)$ does not fork over $X'a'$. Note that $|X| \leq \mathrm{wt}(a'/a)$ and $|X'| \leq \mathrm{wt}(a/a')$.

Let Λ^* be the set of non-orthogonality classes of regular types in $S(M_G)$ of finite dimension in M_G. For each $S \in \Lambda^*$ let (b_s, c_s) be the unique element of $\mathcal{B}_G \times \mathcal{C}_G$ such that there is a candidate (a, b_s, c_s, d, r_d) over a with $r_d \in S$ and let (b'_s, c'_s) be the unique element of $\mathcal{B}'_H \times \mathcal{C}'_H$ such that there is a candidate $(a', b'_s, c'_s, d', r_{d'})$ over a' with $r_{d'} \in S$.

For Λ a finite subset of Λ^*, let $B(\Lambda) = \{b_s : S \in \Lambda\}$, $C(\Lambda) = \{c_s : S \in \Lambda\}$, and $v(\Lambda) = B(\Lambda) \cup C(\Lambda)$. Dually, define $B'(\Lambda)$, $C'(\Lambda)$, and $v'(\Lambda)$ using (b'_s, c'_s) in place of (b_s, c_s).

The proof splits into two cases depending on whether our eni-DOP witness is flexible or inflexible.

Case 1: The eni-DOP witness is inflexible.

This case will be substantially easier than the other, and in fact, we prove that there is a number e such that for all sufficiently large ℓ, the image of any $\ell \times \ell$ bipartite graph contains an $(\ell - e) \times (\ell - e)$ complete, bipartite subgraph. The simplicity of this case is primarily due to the following claim.

Claim 1. For any finite $\Lambda \subseteq \Lambda^*$ such that $v(\Lambda)$ is disjoint from X and $v'(\Lambda)$ is disjoint from X', we have $|v(\Lambda)| = |v'(\Lambda)|$.

Proof. To see this, we again split into cases. First, if $p \perp q$, then we handle the two 'halves' separately. Note that for each $S \in \Lambda$, $\mathrm{tp}(b_s c_s/aa')$ does not fork over a, $\mathrm{tp}(b'_s, c'_s/aa')$ does not fork over a', and by Lemma 4.3,

each element of $\{b_s, c_s\}$ forks with $b'_s c'_s$ over aa'. Since $p \perp q$, this implies $\{b_s, b'_s\}$ fork over aa'. It follows that, working over aa',

$$Cl_p(B(\Lambda)) = Cl_p(B'(\Lambda))$$

hence $|B(\Lambda)| = |B'(\Lambda)|$. It follows by a symmetric argument that $Cl_q(C(\Lambda)) = Cl_q(C'(\Lambda))$, so $|C(\Lambda)| = |C'(\Lambda)|$. It follows immediately that $|v(\Lambda)| = |v'(\Lambda)|$.

On the other hand, if $p \not\perp q$, then Cl_p is a closure relation on $p^*(\mathfrak{C}) \cup q^*(\mathfrak{C})$, where p^* (resp. q^*) is the non-forking extension of p (resp. q) to aa'. Furthermore, for each $S \in \Lambda$ we have $Cl_p(b_s c_s) = Cl_p(b'_s, c'_s)$. It follows that $Cl_p(v(\Lambda)) = Cl_p(v'(\Lambda))$. As each set is independent over aa', we conclude that $|v(\Lambda)| = |v'(\Lambda)|$.

Let $w = \mathrm{wt}(a'/a)$ and $e = w + \mathrm{wt}(a/a')^2$. Suppose that $G_0 \subseteq G$ is an $\ell \times \ell$ complete, bipartite subgraph. Since $|X| \leq w$, there is an $(\ell - w) \times (\ell - w)$ complete subgraph $G_0^* \subseteq G_0$ such that $E_{G_0^*}$ is disjoint from X. By our choice of e there is an $(\ell - e) \times (\ell - e)$ complete subgraph $G_1 \subseteq G_0^*$ such that $\pi_f(b, c)$ is not contained in X' for all pairs $(b, c) \in E_{G_1}$. But then, by Lemma 4.3, we have $\pi_f(b, c)$ is disjoint from X' for all $(b, c) \in E_{G_1}$.

Now, G_1 is an $(\ell - e) \times (\ell - e)$ complete, bipartite subgraph of G. In particular, G_1 has $2(\ell - e)$ vertices and $(\ell - e)^2$ edges. Let H_1 be the subgraph of H whose edges are $E_{H_1} := \pi_f(E_{G_1})$ and whose vertices are $v(H_1) := v_H(E_{H_1})$. Then $|E_{H_1}| = (\ell - e)^2$ since π_f is a bijection and

$$|v(H_1)| = |v_H(E_{H_1})| = |v_G(E_{G_1})| = 2(\ell - e)$$

by Claim 1. By a classical optimal packing result, this is only possible when H_1 is itself a complete, $(\ell - e) \times (\ell - e)$ bipartite subgraph of H.

Case 2: The eni-DOP witness is flexible.

As we insisted that our finite approximation be flexible, it follows from Lemma 4.1 that $p \not\perp q$, so p-closure is a dependence relation on $p(\mathfrak{C}) \cup q(\mathfrak{C})$.

As well, for any candidate (b, c, d, r_d) over a and for any finite $A \supseteq a$, there is an equivalent candidate $(b', c', d', r_{d'})$ over a such that $w_p(b'c'/A) = 1$.

Definition 4.5 For any finite subgraph $G_0 \subseteq G$, let $\Lambda(G_0)$ be the set of non-orthogonality classes $\{[r_{d_{g,h}}] : (g, h) \in E_{G_0}\}$. Note that $|\Lambda(G_0)| = |E_{G_0}|$ by the pairwise orthogonality of the types $r_{d_{g,h}}$.

A *manifestation* $\mathcal{M} = \mathcal{M}(\Lambda, a)$ *over* a is a set of candidates $\{(b_s, c_s, d_s, r_{d_s}) : S \in \Lambda\}$ over a with $r_{d_s} \in S$ for each $S \in \Lambda$. Associated to any manifestation

$\mathcal{M}$ is a bipartite graph $G(\mathcal{M})$ with 'Left Nodes' $L(\mathcal{M}) = \{b_s : s \in \Lambda\}$, 'Right Nodes' $R(\mathcal{M}) = \{c_s : s \in \Lambda\}$, vertices $v(\mathcal{M}) = L(\mathcal{M}) \cup R(\mathcal{M})$, and edges $E(\mathcal{M}) = \{(b_s, c_s) : s \in \Lambda\}$.

If G_0 is a subgraph of G, then the *canonical manifestation of* $\Lambda(G_0)$ *over* a *inside* M_G is the set

$$\{(b_g, c_h, d_{g,h}, r_{g,h}) : (g,h) \in E_{G_0}\}$$

.

A set A *represents* Λ *over* a if $a \subseteq A$ and $v(\mathcal{M}) \subseteq A$ for some manifestation $\mathcal{M}$ of Λ over a. A manifestation $\mathcal{M}'$ is A-*free* if $w_p(b'_s, c'_s/A) = 1$ for each $S \in \Lambda$ and $\{(b'_s, c'_s) : S \in \Lambda\}$ are independent over A.

Now, working in the monster model $\mathfrak{C}$, we define a measure of the complexity of Λ over a. First, note that for any candidate (b, c, d, r_d) over a, there is an equivalent candidate $(b','c', d', r_{d'})$ over a with $w_p(b'c'/abc) = 1$. By choosing $b'c'$ to be independent over abc from any given $A \supseteq abc$ we can insist that $w_p(b'c'/A) = 1$. It follows that A-free manifestations of Λ exist over any set A representing a finite Λ. Thus, the following definition makes sense.

Definition 4.6 The *maximal weight,* $mw(\Lambda, a)$, is the largest integer m such that for all finite A representing Λ over a, there is an A-free manifestation $\mathcal{M}'(\Lambda, a)$ over a with $|v(\mathcal{M}')| = m + \Lambda$.

Lemma 4.7 *Suppose that G is a bipartite graph, $G_0 \subseteq G$ is a connected subgraph of G, let $\mathcal{M}(\Lambda(G_0), a)$ be the canonical manifestation of $\Lambda(G_0)$ inside M_G, and let $\mathcal{M}'(\Lambda, a)$ be any other manifestation of $\Lambda(G_0)$. Then*

$$Cl_p(v(\mathcal{M}') \cup \{v\}) = Cl_p(v(\mathcal{M}') \cup v(G_0))$$

for any $v \in v(G_0)$.

Proof. Arguing by symmetry and induction, it suffices to show that for all nonempty $B \subseteq v(G_0)$ and every $c \in v(G_0) \setminus B$ such that $(b, c) \in E_{G_0}$ for some $b \in B$ we have that $c \in Cl_p(v(\mathcal{M}') \cup B)$. But this is immediate, since $Cl_p(\{b', c', b, c\}) = Cl_p(\{b', c', b\})$ for all equivalent candidates (b, c, d, r_d) and $(b', c', d', r_{d'})$ over a.

Lemma 4.8 $k(G_0) \leq mw(\Lambda(G_0), a) \leq |v(G_0)|$ *for any bipartite graph* G *and any finite* $G_0 \subseteq G$.

Proof. The upper bound is very soft. Let $A \supseteq a \cup v(G_0)$ be arbitrary and let $\mathcal{M}'$ be any other manifestation of $\Lambda(G_0)$ over a. Then

$$w_p(v(\mathcal{M}')/a) \leq w_p(v(\mathcal{M}')v(G_0)/a) = w_p(v(\mathcal{M}')/av(G_0)) + w_p(v(G_0)/a)$$

Since $w_p(b'_s c'_s/ab_s c_s) \leq 1$ for each $S \in \Lambda(G_0)$, we have $w_p(v(\mathcal{M}')/av(G_0)) \leq |\Lambda(G_0)|$. Also, by the independence of the nodes in M_G, $w_p(v(G_0)/a) = |v(G_0)|$. The upper bound on $mw(\Lambda(G_0), a)$ follows immediately.

For the lower bound, again choose any $A \supseteq av(G_0)$ and let $C \subseteq v(G_0)$ consist of one vertex from every connected component of G_0. Clearly, A represents $\Lambda(G_0)$ and $|C| = CC(G_0)$. Let $\mathcal{M}'$ be any A-free manifestation of $\Lambda(G_0)$ over a. Then

$$w_p(v(\mathcal{M}')/a) \geq w_p(v(\mathcal{M}')C/a) - CC(G_0) = w_p(v(\mathcal{M}')v(G_0)/a) - CC(G_0)$$

with the second equality coming from Lemma 4.7. As before, for each $S \in \Lambda(G_0)$, $w_p(b'_s c'_s/ab_s c_s) \leq 1$ so $w_p(v(\mathcal{M}')/av(G_0)) \leq |\Lambda(G_0)|$. On the other hand, the A-freeness of $\mathcal{M}'$ implies that $w_p(v(\mathcal{M}')/A) = |\Lambda(G_0)|$, hence $w_p(v(\mathcal{M}')/av(G_0)) = |\Lambda(G_0)|$. Thus,

$$w_p(v(\mathcal{M}')v(G_0)/a) = w_p(v(\mathcal{M}')/v(G_0)a) + w_p(v(G_0)/a) = |\Lambda(G_0)| + |v(G_0)|$$

from which the lower bound follows as well.

Now, returning to our isomorphism $f : M_G \to M_H$, suppose that G_0 is any finite subgraph of G that is disjoint from X, i.e., so that $\mathrm{tp}(G_0/aa')$ does not fork over a. We then claim:

Claim 2: $mw(\Lambda(G_0), a') \leq |v(G_0)| + wt(a/a')$

Proof. Choose any finite A containing $\{aa'\} \cup v(G_0) \cup v_H(\pi_f(E_{G_0}))$. So A represents $\Lambda(G_0)$ over a'. Let $\mathcal{M}'$ be any A-free manifestation of $\Lambda(G_0)$ over a'. Now

$$w_p(v(\mathcal{M}')/a') \leq w_p(v(\mathcal{M}')aG_0/a') = w_p(v(\mathcal{M}')/aa'G_0) + w_p(aG_0/a')$$

But, as before $w_p(b'_s c'_s/aa'b_s c_s) \leq 1$, so $w_p(v(\mathcal{M}')/aa'G_0) \leq |\Lambda(G_0)|$. Also,

$$w_p(aG_0/a') = w_p(G_0/aa') + wt(a/a') = |v(G_0)| + w_p(a/a')$$

and the Claim follows.

Finally, choose a complete, bipartite subgraph $G_0 \subseteq G$, where ℓ is sufficiently large with respect to $W = wt(a/a')$. Let H_0 be the bipartite graph with vertices $v_H(\pi_f(E_{G_0}))$ and edges $\pi_f(E_{G_0})$ and let H_0^* be the subgraph of H with the same vertex set as H_0. Note that $E_{H_0} \subseteq E_{H_0^*}$, but that equality need not hold.

As G_0 is $\ell \times \ell$ complete bipartite, $|v(G_0)| = 2\ell$ and $|\Lambda(G_0)| = \ell^2$. It follows immediately that $|E_{H_0}| = \ell^2$ and it follows from Claim 2 and Lemma 4.8 that

$$k(H_0) \leq mw(\Gamma(G_0), a') \leq 2\ell + W$$

where $W = wt(a/a')$. So, by Corollary A.7 of the Appendix, H_0 contains an almost ℓ-complete bipartite subgraph H_1. But then, H_1^*, which is the subgraph of H with the same vertex set as H_1, is almost ℓ-complete as well.

4.3 Coding trees by complete, bipartite subgraphs

As Proposition 4.4 is rather weak, we give up on coding arbitrary bipartite graphs into models of T. Rather, we seek to code subtrees of $\lambda^{<\omega}$ into bipartite graphs that have large, complete subgraphs.

Fix a sufficiently large integer m and a tree $\mathcal{T} \subseteq \lambda^{<\omega}$. We will construct a bipartite graph $G_{\mathcal{T}}^{[m]}$, whose $7m \times 7m$ complete bipartite subgraphs $B_{\mathcal{T}}^m(\eta)$ code nodes $\eta \in \mathcal{T}$. Moreover, additional information about the level of η and its set of immediate successors will be coded by the size of the intersection of $B_{\mathcal{T}}^m(\eta)$ and $B_{\mathcal{T}}^m(\nu)$ for other $\nu \in T$.

More precisely, fix a tree $(\mathcal{T}, \trianglelefteq)$ and a large integer m. We first define a bipartite graph $preG_{\mathcal{T}}^{[m]}$ to have universe $\mathcal{T} \times m \times 14$ with the edge relation

$$\{((\eta, i_1, n_1), (\eta, i_2, n_2)) : \eta \in \mathcal{T}, i_i, i_2 \in m, n_1 + n_2 \text{ is odd}\}$$

So the 'left hand side' of $preG_{\mathcal{T}}^{[m]}$ is $L = \mathcal{T} \times m \times \{n \in 14 : n \text{ odd}\}$, the 'right hand side' is $R = \mathcal{T} \times m \times \{n \in 14 : n \text{ even}\}$, thereby associating a $7m \times 7m$ complete, bipartite graph to each node $\eta \in \mathcal{T}$.

Next, define a binary relation E_0 on $preG_{\mathcal{T}}^{[m]}$ by $(\eta_1, i_1, n_1)E_0(\eta_2, i_2, n_2)$ if and only if

- η_2 is an immediate successor of η_1, $i_1 = i_2$, $n_1 = n_2$ and

- either $\lg(\eta_1) = 0$ and $n_1 \in \{0, 1\}$ or $\lg(\eta_1) > 0$ and $n_1 \in \{10, 11, 12, 13\}$.

Let E be the smallest equivalence relation containing E_0, i.e., the reflexive, symmetric and transitive closure of E_0.

Let $G^{[m]}_{\mathcal{T}} := preG^{[m]}_{\mathcal{T}}/E$ and, for each $\eta \in \mathcal{T}$, let $B^m_{\mathcal{T}}(\eta) = \{g \in G^{[m]}_{\mathcal{T}} : (\eta, i, n) \in g$ for some $i < m, n < 14\}$.

As notation, for each $\eta \in \mathcal{T}$, let $B^m_{\mathcal{T}}(\eta) = \{g \in G^{[m]}_{\mathcal{T}} : (\eta, i, n) \in g$ for some $i < m, n < 14\}$, let $\mathcal{S}^m_{\mathcal{T}} = \{B^m_{\mathcal{T}}(\eta) : \eta \in \mathcal{T}\}$, and let $g_{\mathcal{T}} : \mathcal{T} \to \mathcal{S}^m_{\mathcal{T}}$ be the bijection $\eta \mapsto B^m_{\mathcal{T}}(\eta)$. For all of these, when $\mathcal{T}$ and m are clear, we delete reference to them. Finally, call an element $g \in G^{[m]}_{\mathcal{T}}$ a *singleton* if $g = \{(\eta, i, n)\}$ for a single element $(\eta, i, n) \in preG^{[m]}_{\mathcal{T}}$. All of the following Facts are immediate:

Fact 4.9 1. *Every $B(\eta)$ is a $7m \times 7m$ complete, bipartite graph;*

2. *If $g \in B(\eta)$ is a singleton and $E(g, h)$, then $h \in B(\eta)$;*

3. *For all $\eta \in \mathcal{T}$, $i < m$, (η, i, n) is a singleton for all $2 \leq n \leq 9$.*

4. *If $lg(\nu) < lg(\eta)$, then $B(\nu) \cap B(\eta) = \emptyset$ if and only if $\nu = \eta^-$. Moreover, a nonempty intersection is a complete $m \times m$ bipartite graph if $\eta^- = \langle\rangle_{\mathcal{T}}$ and the intersection is $2m \times 2m$ complete, bipartite if $\eta^- \neq \langle\rangle_{\mathcal{T}}$.*

Lemma 4.10 $\mathcal{S} = \{$*all $7m \times 7m$ complete, bipartite subgraphs of* $G^{[m]}_{\mathcal{T}}\}$.

Proof. That each $B(\eta) \in \mathcal{S}$ is a $7m \times 7m$ complete, bipartite subgraph is clear. Conversely, fix a $7m \times 7m$ complete, bipartite subgraph of $G^{[m]}_{\mathcal{T}}$. First, suppose that X contains a singleton a. Without loss, assume $a \in X \cap B(\eta) \cap L$. Then $E_X(a) = \{b \in X : E(a, b)\}$ has cardinality $7m$ and is contained in $B(\eta) \cap R$, hence $E_X(a) = B(\eta) \cap R$. But then, $X \cap R$ contains a singleton as well, so arguing similarly, $B(\eta) \cap L = X \cap L$, so $X = B(\eta)$. It remains to show that X contains a singleton. Choose k maximal such that there is $\eta \in \mathcal{T}$, $lg(\eta) = n$, and $X \cap B(\eta) \neq \emptyset$. Let $\nu = \eta^-$. If X does not contain a singleton, then the maximality of k implies that $X \cap (B(\eta') \setminus B(\nu)) = \emptyset$ for all $\eta' \in Succ(\nu)$. Choose any $a \in X \cap B(\eta) \cap L$. Then $a \in B(\nu)$ and moreover, $E_X(a) \subseteq B(\nu) \cap R$. By counting, $E_X(a) = B(\nu) \cap R$, so X contains a singleton, completing the proof of the Claim.

For clarity, let $L_0 = \{R_1, R_2\}$ denote the language consisting of two binary relation symbols. Form an L_0 structure $(\mathcal{S}^m_{\mathcal{T}}, R_1, R_2)$ by positing that $R_1(X,Y)$ holds if and only if $X \cap Y$ is an $m \times m$ complete, bipartite graph and $R_2(X,Y)$ holds if and only if $X \cap Y$ is a $2m \times 2m$ complete, bipartite graph.

Lemma 4.11 *For any sufficiently large m and trees $(\mathcal{T}, \trianglelefteq), (\mathcal{T}', \trianglelefteq)$, if there is an L_0-isomorphism $\Phi : (\mathcal{S}^m_{\mathcal{T}}, R_1, R_2) \to (\mathcal{S}^m_{\mathcal{T}'}, R_1, R_2)$ of the associated L_0-structures, then the composition $h : (\mathcal{T}, \trianglelefteq) \to (\mathcal{T}', \trianglelefteq)$ given by $h = g^{-1}_{\mathcal{T}'} \circ \Phi \circ g_{\mathcal{T}}$ is a tree isomorphism.*

Proof. For each $n \in \omega$, let $\mathcal{T}_n = \{\eta \in \mathcal{T} : lg(\eta) < n\}$ and define $\mathcal{T}'_n$ analogously. Using Fact 4.9(4), one proves by induction on n that $h|_{\mathcal{T}_n} : (\mathcal{T}_n, \trianglelefteq) \to (\mathcal{T}'_n, \trianglelefteq)$ is a tree isomorphism. This suffices to prove the Lemma.

4.4 $\aleph_0$-stable, eni-DOP theories are λ-Borel complete

Theorem 4.12 *If T is $\aleph_0$-stable with eni-DOP, then for any infinite cardinal λ, there is a λ-Borel embedding $\mathcal{T} \mapsto M(\mathcal{T})$ from subtrees of $\lambda^{<\omega}$ to $Mod_\lambda(T)$ satisfying*

$$(\mathcal{T}_1, \trianglelefteq) \cong (\mathcal{T}_2, \trianglelefteq) \qquad \textit{if and only if} \qquad M(\mathcal{T}_1) \cong M(\mathcal{T}_2)$$

Proof. Fix any infinite cardinal λ. As in Subsection 4.2, fix an eni-DOP witness (M_0, M_1, M_2, M_3, r) and a finite approximation $\mathcal{F} = (a, b, c, d, r_d)$ of it, choosing $\mathcal{F}$ to be flexible if the witness is. As notation, let $p = \mathrm{tp}(b/a)$ and $q = \mathrm{tp}(c/a)$. As well, for the whole of the proof, fix a recursive, fast growing sequence, $\langle m_i : i \in \omega \rangle$ of integers, e.g., $m_0 = 10$ and $m_{i+1} = m_i!!$

Given a subtree $\mathcal{T} \subseteq \lambda^{<\omega}$, let $G^*_{\mathcal{T}}$ be the bipartite graph which is the disjoint union $\bigcup_{i\in\omega} G^{[m_i]}_{\mathcal{T}}$, where the graphs $G^{[m_i]}_{\mathcal{T}}$ are constructed as in Subsection 4.3. Next, construct a model $M(\mathcal{T}) := M_{G^*_{\mathcal{T}}}$ from the bipartite graph $G^*_{\mathcal{T}}$ as in Subsection 4.2. Clearly, after some reasonable coding, we may assume that $M(\mathcal{T})$ has universe λ. It is routine to verify that both of the maps $\mathcal{T} \mapsto G^*_{\mathcal{T}}$ and $G^*_{\mathcal{T}} \mapsto M_{G^*_{\mathcal{T}}}$ (and hence their composition) are λ-Borel.

By looking at the constructions in Subsections 4.2 and 4.3, it is easily checked isomorphic trees $\mathcal{T} \cong \mathcal{T}'$ give rise to isomorphic models $M(\mathcal{T}) \cong M(\mathcal{T}')$.

To establish the converse, suppose that $\mathcal{T}, \mathcal{T}'$ are subtrees such that $M(\mathcal{T}) \cong M(\mathcal{T}')$. Fix an isomorphism $f : M_{G^*_{\mathcal{T}}} \to M_{G^*_{\mathcal{T}'}}$ and choose i so that $m_i >> \ell^*$, where ℓ^* is the constant in the statement of Proposition 4.4.

For each $\eta \in \mathcal{T}$, by Fact 4.9(1), $B^m_{\mathcal{T}}(\eta)$ is a $7m_i \times 7m_i$ complete, bipartite subgraph of $G^{[m_i]}_{\mathcal{T}}$. Let $E(\eta)$ denote the edge set of $B^m_{\mathcal{T}}(\eta)$. Now $\pi_f(E(\eta))$ is a set of $(7m_i)^2$ edges in $G^*_{\mathcal{T}'}$. Let $v_{\mathcal{T}'}(\eta)$ be the smallest set of vertices in $G^*_{\mathcal{T}'}$ whose edge set contains $\pi_f(E(\eta))$.

By Proposition 4.4, the graph $J(\eta) := (v_{T'}(\eta), \pi_f(E(\eta)))$ has an almost $7m_i$-complete bipartite subgraph $K(\eta)$. Let $K^*(\eta)$ be the subgraph of $G^*_{\mathcal{T}'}$ whose vertex set is the same as $K(\eta)$. Note that the edge set of $K^*(\eta)$ contains the edge set of $K(\eta)$, so $K^*(\eta)$ is almost $7m_i$-complete as well.

As $K^*(\eta)$ is a connected subgraph of $G^*_{\mathcal{T}'}$, $K^*(\eta) \subseteq G^{[m_j]}_{\mathcal{T}'}$ for some j. As the valence of each vertex of $K^*(\eta)$ is $\sim 7m_i$ and $m_i >> m_k$ for all $k < i$, we must have $j \geq i$.

Claim: $j = i$.

Proof. Choose $\nu \in \mathcal{T}'$ such that $K^*(\eta)$ and $B^{m_j}(\nu)$ share a connected subgraph D with $e(D) >> N_f$. Arguing as above, there is an almost $7m_j$ complete, bipartite subgraph $H^*(\nu)$ of $G^*_{\mathcal{T}}$ whose edge set (almost) contains $\pi_f^{-1}(E(\nu))$, where $E(\nu)$ is the edge set of $B^{m_j}_{\mathcal{T}'}(\nu)$. As before, $H^*(\nu) \subseteq G^{[m_k]}_{\mathcal{T}}$ for some k, and as the valence of every vertex is large, $k \geq j$. However, almost all of the edges of D correspond to edges of $H^*(\nu)$. In particular, $H^*(\nu)$ contains edges from $B^{m_i}_{\mathcal{T}}(\eta)$. But, as $H^*(\eta)$ is connected, this implies $H^*(\eta) \subseteq G^{[m_i]}_{\mathcal{T}}$. Thus $k = j = i$.

Thus, we have shown that for each $\eta \in \mathcal{T}$, $K^*(\eta)$ is an almost $7m_i$ complete bipartite subgraph of $G^{[m_i]}_{\mathcal{T}'}$. It follows as in the proof of Lemma 4.10 that for each $\eta \in \mathcal{T}'$, there is a unique $\nu \in \mathcal{T}'$ such that the subgraphs $K^*(\eta)$ and $B^{m_i}_{\mathcal{T}'}(\nu)$ have large intersection in $G^{[m_i]}_{\mathcal{T}'}$. Define

$$\Phi : \mathcal{S}^{m_i}_{\mathcal{T}} \to \mathcal{S}^{m_i}_{\mathcal{T}'}$$

by $\Phi(B^{m_i}(\eta)) = B^{m_i}_{\mathcal{T}'}(\nu)$ for this unique ν. As the argument given above is reversible, Φ is a bijection. Furthermore, if $D \subseteq B^{m_i}(\eta)$ is either an $m \times m$ or a $2m \times 2m$ complete, bipartite subgraph, then applying Proposition 4.4 to D yields a connected graph $K^*(D)$ whose number of edges satisfies

$$m_i^2 - N_f \leq e(K(D)) \leq m_i^2$$

By taking $D = B^{m_i}(\eta_1) \cup B^{m_i}(\eta_2)$ for various $\eta_1, \eta_2 \in \mathcal{T}$, it follows that Φ is an L_0-isomorphism. Thus, by Lemma 4.11, $(\mathcal{T}, \trianglelefteq) \cong (\mathcal{T}', \trianglelefteq)$ as required.

Corollary 4.13 *If T is $\aleph_0$-stable with eni-DOP, then T is Borel complete. Moreover, for every infinite cardinal λ, T is λ-Borel complete for $\equiv_{\infty,\aleph_0}$.*

Proof. For both statements, by Theorem 3.11, it suffices to show that

$$(\text{Subtrees of } \lambda^{<\omega}, \equiv_{\infty,\aleph_0}) \ \leq^B_\lambda \ (Mod_\lambda(T), \equiv_{\infty,\aleph_0})$$

for every $\lambda \geq \aleph_0$. So fix an infinite cardinal λ. The map $\mathcal{T} \mapsto M(T)$ given in Theorem 4.12 is λ-Borel. Choose any generic filter G for the Levy collapsing poset $Lev(\lambda, \aleph_0)$. By Fact 3.5, for any subtrees $\mathcal{T}_1, \mathcal{T}_2 \subseteq \lambda^{<\omega}$ in V, $\mathcal{T}_1 \equiv_{\infty,\aleph_0} \mathcal{T}_2$ in V if and only if $\mathcal{T}_1 \cong \mathcal{T}_2$ in $V[G]$. As well, $M(\mathcal{T}_1) \equiv_{\infty,\aleph_0} M(\mathcal{T}_2)$ in V if and only if $M(\mathcal{T}_1) \cong M(\mathcal{T}_2)$ in $V[G]$. Thus, since the mapping $\mathcal{T} \mapsto M(\mathcal{T})$ is visibly absolute between V and $V[G]$, the result follows immediately from Theorem 4.12.

5 eni-NDOP and decompositions of models

In this section, we assume throughout that T is $\aleph_0$-stable with eni-NDOP. [In fact, the first few Lemmas require only $\aleph_0$-stability.] We discuss three species of decompositions (regular, eni, and eni-active) of an arbitrary model M and prove a theorem about each one. Theorem 5.10 asserts that in a regular decomposition $\mathfrak{d} = \langle M_\eta, a_\eta : \eta \in I \rangle$ of M, then M is atomic over $\bigcup_{\eta \in I} M_\eta$. This theorem plays a key role in Corollary 5.19.

Next, we discuss eni-active decompositions of a model M and prove that for any $N \preceq M$ that contains $\bigcup_{\eta \in I} M_\eta$, then N is an $\infty, \aleph_0$-elementary substructure of M. In particular, Corollary 5.16 states that an eni-active decomposition determines a model up of $\equiv_{\infty,\aleph_0}$-equivalence. This is extremely important when we compute $I_{\infty,\aleph_0}(T, \kappa)$.

Finally, we prove Theorem 5.18, which states that a model M is atomic over $\bigcup_{\eta \in I} M_\eta$ for any eni decomposition of M *provided that each of the models is maximal atomic* (see Definition 5.17). While the result sounds strong, it is of little use to us, as one has little control about what the maximal atomic submodels of an arbitrary model look like. This theorem was also proved by Koerwien [3], but is included here to contrast with Theorems 5.10 and 5.15.

We begin with some Lemmas that are implicit in [11].

Lemma 5.1 *Suppose that $A \subseteq B$ are sets, $p = \mathrm{tp}(c/B)$ is stationary and isolated by $\varphi(x, b)$, and $p \perp A$. Then $\varphi(x, b)$ isolates $\mathrm{tp}(c/BD)$ for any set D satisfying $D \underset{A}{\perp} B$.*

Proof. Choose any d from D and let $q = stp(d/A)$. As $p \perp A$, $p \perp q$, hence $c' \underset{B}{\perp} d$ for any c' realizing p. That is, $\mathrm{tp}(c/B) = \mathrm{tp}(c'/B)$ implies $\mathrm{tp}(cd/B) = \mathrm{tp}(c'd/B)$, which implies $\mathrm{tp}(c/Bd) = \mathrm{tp}(c'/Bd)$. So $\mathrm{tp}(c/B) \vdash \mathrm{tp}(c/D)$ and the result follows.

Lemma 5.2 *Suppose that $A \underset{E}{\perp} B$, $E = A \cap B$, N is atomic over $A \cup B$, $p \in S(N)$ is regular but not eni, $p \perp A$, and $p \perp B$. Then $N \cup \{c\}$ is atomic over $A \cup B$ for any c realizing p.*

Proof. It suffices to show that for every finite $D \subseteq N$, there is a finite D', $D \subseteq D' \subseteq N$ such that $\mathrm{tp}(c/D'AB)$ is isolated. Fix a finite $D \subseteq N$. First, choose a finite $D_1 \supseteq D$ over which p is based and stationary. Next, choose $X \subseteq B$ finite such that $D_1 \underset{EXA}{\perp} B$ and let $D_2 = D_1X$. Note that $B \underset{EX}{\perp} AD_2$ and EX is a subset of both AD_2 and B. Finally, choose a finite $Y \subseteq A$ such that $D_2 \underset{Y}{\perp} A$ and let $D' = D_2Y$. Note that $D'A = D_2A$, so $B \underset{EX}{\perp} D'A$ and $EX \subseteq D'A$.

Since D' is finite and p is not eni, $\mathrm{tp}(c/D')$ is isolated and parallel to p. However, $Y \subseteq A$, so $p \perp Y$, and $D' \underset{Y}{\perp} A$. So, by Lemma 5.1, $\mathrm{tp}(c/D'A)$ is isolated and parallel to p as well. Similarly, $EX \subseteq B$, so $p \perp EX$ and $D'A \underset{EX}{\perp} B$. Thus, $\mathrm{tp}(c/D'AB)$ is isolated by another application of Lemma 5.1.

Lemma 5.3 *Suppose $A \cup F \subseteq B$, $p \in S(B)$ is stationary, regular, but not eni, F finite, and $p \perp A$. If B is atomic over $A \cup F$, then $B \cup \{c\}$ is atomic over $A \cup F$ for any c realizing p.*

Proof. Choose any realization c of p. It suffices to prove that for any finite D such that $F \subseteq D \subseteq B$, there is a finite D', $D \subseteq D' \subseteq B$ such that $D' \cup \{c\}$ is atomic over $A \cup F$. However, to achieve this, it suffices to show that $\mathrm{tp}(c/D' \cup A)$ is isolated.

So fix any finite D with $F \subseteq D \subseteq B$. Choose a finite D' such that $D \subseteq D' \subseteq B$, p is based and stationary over D', and $\mathrm{tp}(D'/A)$ does not

fork over some finite $E \subseteq D' \cap A$. Since p is not eni, $\text{tp}(c/D')$ is isolated. But $p \perp A$ implies $p \perp E$, so $p|D'$ is almost orthogonal to $\text{tp}(A/E)$. Thus, $\text{tp}(c/D') \vdash \text{tp}(c/D' \cup A)$, so $\text{tp}(c/D' \cup A)$ is isolated as well.

Lemma 5.4 *Suppose that J is an independent set over A, $B \supseteq A \cup J$ is atomic over $A \cup J$, and $p = \text{tp}(a/B)$ is regular, not eni, and orthogonal to A. Then $B \cup \{a\}$ is atomic over $A \cup J$.*

Proof. Fix any finite $B_0 \subseteq B$. It suffices to find a finite D, $B_0 \subseteq D \subseteq B$ such that $\text{tp}(a/D)$ is isolated and $\text{tp}(a/D) \vdash \text{tp}(a/DAJ)$. To find such a D, first choose $B_1 \supseteq B_0$ on which p is based and stationary. Next, choose a finite $J_0 \subseteq J$ such that, letting $J_1 = J \setminus J_0$, we have $B_1 \underset{J_0 A}{\perp} J_1$. Note that by the A-independence of J, this implies

$$B_1 J_0 \underset{A}{\perp} J_1$$

Finally, choose $E \subseteq A$ such that $B_1 J_0 \underset{E}{\perp} A$ and let $D = B_1 \cup J_0 \cup E$. As p is not eni, $\text{tp}(a/D)$ is isolated. In addition, since $E \subseteq A$, $p \perp E$, so $\text{tp}(a/D) \vdash \text{tp}(a/DA)$ by Lemma 5.1. As well, since $p \perp A$ and $D \underset{A}{\perp} J_1$, a second application of Lemma 5.1 implies $\text{tp}(a/DA) \vdash \text{tp}(a/DAJ)$.

Lemma 5.5 *For any nonzero limit ordinal α, suppose $\langle B_i : i < \alpha \rangle$ is a continuous, increasing sequence of sets such that for each i, there is a model $M_i \subseteq B_i$ such that $B_i \underset{M_i}{\perp} (B_{i+1} \setminus B_i)$. Suppose that $\langle A_i : i < \alpha \rangle$ is a continuous, increasing sequence of subsets of a given model N such that each A_i contains B_i and is atomic over B_i. Then, letting $A = \bigcup_{i<\alpha} A_i$ and $B = \bigcup_{i<\alpha} B_i$, each $B_i \subseteq_{TV} B$ and A is atomic over B. In fact, if each A_i is maximal atomic over B_i, then A is maximal atomic over B.*

Proof. That $B_i \subseteq_{TV} B_{i+1}$ is Lemma 1.9(2), and the preservation of the TV-property under continuous chains of sets is identical to the preservation of elementarity under continuous chains of models. To see that A is atomic over B, choose a finite subset $D \subseteq A$ and choose i such that $D \subseteq A_i$. If $\varphi(x, b_i)$ isolates $\text{tp}(D/B_i)$, then the same formula isolates $\text{tp}(D/B)$, hence A is atomic over B.

To obtain the final sentence, suppose that each A_i is maximal atomic over B_i. By $\aleph_0$-stability, each A_i is a model, so A is a model as well. Assume by way of contradiction that there is $c \in N \setminus A$ such that Ac is atomic over B. Choose $i < \alpha$ such that $\mathrm{tp}(c/A)$ does not fork and is stationary over A_i. We will obtain a contradiction by showing that cA_i is atomic over B_i. By iterating Lemma 1.9(2), $A_i \underset{B_i}{\perp} B$, so $cA_i \underset{B_i}{\perp} B$. Choose a finite $D \subseteq A_i$. As cA was assumed to be atomic over B, $\mathrm{tp}(cD/B)$ is isolated. By the Open Mapping Theorem, this implies $\mathrm{tp}(cD/B_i)$ is isolated, which implies cA_i is atomic over B_i.

With the preliminaries out of the way, we introduce various species of decompositions inside and of a model M.

Definition 5.6 An *independent tree of models* $\{M_\eta : \eta \in I\}$ satisfies

- I is a subtree of $Ord^{<\omega}$;
- $\eta \trianglelefteq \nu$ implies $M_\eta \preceq M_\nu$;
- For each $\eta \neq \langle\rangle$ and each $\nu \in Succ_I(\eta)$, $\mathrm{tp}(M_\nu/M_\eta) \perp M_{\eta^-}$; and
- For each $\eta \in I$ and $\nu \in Succ_I(\eta)$, $\bigcup_{\nu \trianglelefteq \gamma} M_\gamma \underset{M_\eta}{\perp} \bigcup_{\nu \not\trianglelefteq \delta} M_\delta$

We define a family of notions of decompositions.

Definition 5.7 Fix a model M. A *[regular, eni, eni-active] decomposition inside M* $\mathfrak{d} = \langle M_\eta, a_\eta : \eta \in I\rangle$ consists of an independent tree $\{M_\eta : \eta \in I\}$ of elementary submodels of M indexed by $(I, \trianglelefteq)$ satisfying the following conditions for each $\eta \in I$:

1. Each $a_\eta \in M_\eta$ (but $a_{\langle\rangle}$ is meaningless);
2. The set $C_\eta := \{a_\nu : \nu \in Succ_I(\eta)\}$ is independent over M_η;
3. For each $\nu \in Succ_I(\eta)$ we have:
   (a) $\mathrm{tp}(a_\nu/M_{\nu^-})$ is [regular, eni, eni-active];
   (b) If $\eta \neq \langle\rangle$, then $\mathrm{tp}(a_\nu/M_\eta) \perp M_{\eta^-}$;
   (c) M_ν is atomic over $M_\eta \cup \{a_\nu\}$;

A *[regular, eni, eni-active] decomposition of* M is a [regular, eni, eni-active] decomposition inside M with the additional property that for each $\eta \in I$, the set C_η is a **maximal** M_η-independent set of realizations of [regular, eni, eni-active] types (that are orthogonal to M_{η^-} when $\eta \neq \langle\rangle$).

We say that a decomposition (of any sort) is *prime* if $M_{\langle\rangle}$ is a prime submodel of M and, for each $\nu \neq \langle\rangle$, M_ν is prime over $M_{\nu^-} \cup \{a_\nu\}$.

It is important to note that even though eni-NDOP implies eni-active NDOP, it is not the case that every eni-active decomposition is an eni decomposition. It is also useful to recognize that in any eni-active decomposition, for every maximal branch $B \subseteq I$, $\langle M_b, a_b : b \in B\rangle$ is an eni-active chain (see Definition 1.2).

As well, note that if $\mathfrak{d} = \langle M_\eta, a_\eta : \eta \in I\rangle$ is a decomposition of M (in any of the senses) and $N \preceq M$ contains $\bigcup_{\eta\in I} M_\eta$, then $\mathfrak{d}$ is also a decomposition of N.

The following Lemma requires no assumptions beyond $\aleph_0$-stability.

Lemma 5.8 *For any M, prime [regular, eni, eni-active] decompositions of M exist.*

Proof. Simply start with an arbitrary prime model $M_{\langle\rangle} \preceq M$, and given a node M_η, choose C_η to be any maximal M_η-independent subset of M of realizations of [regular, eni, eni-active] types (that are orthogonal to M_{η^-} when $\eta \neq \langle\rangle$) and, for each $a_\nu \in C_\eta$, choose $M_\nu \preceq M$ to be prime over $M_\eta \cup \{a_\nu\}$. Any maximal construction of this sort will produce a prime [regular, eni, eni-active] decomposition of M.

Of course, without any additional assumptions, such a decomposition may be of limited utility.

Until the end of this section, we assume T is $\aleph_0$-stable with eni-NDOP.

Lemma 5.9 *Let $\mathfrak{d} = \langle M_\eta, a_\eta : \eta \in I\rangle$ be any regular decomposition inside $\mathfrak{C}$ and let N be atomic over $\bigcup_{\eta\in I} M_\eta$. If $p \not\perp N$, then $p \not\perp M_\eta$ for some $\eta \in I$.*

Proof. Recall that eni-NDOP implies eni-active NDOP by Theorem 2.4. Choose a finite $E \subseteq N$ over which p is based and stationary. As E is finite and atomic over $\bigcup_{\eta\in I} M_\eta$, we can find a finite subtree $J \subseteq I$ such that E is

atomic over $\bigcup_{\eta \in J} M_\eta$. Fix such a J and choose $M_J \preceq N$ such that $E \subseteq M_J$ and M_J is prime over $\bigcup_{\eta \in J} M_\eta$. As $E \subseteq M_J$, $p \not\perp M_J$, so by eni-NDOP there is an $\eta \in J$ such that $p \not\perp M_\eta$.

Theorem 5.10 (T $\aleph_0$-stable with eni-NDOP) *Suppose $\langle M_\eta, a_\eta : \eta \in I \rangle$ is a regular decomposition of M. Then M is atomic over $\bigcup_{\eta \in I} M_\eta$.*

Proof. Choose an enumeration $\langle \eta_i : i \in \alpha \rangle$ of I such that $\eta_i \trianglelefteq \eta_j$ implies $i \leq j$. As notation, for $i < \alpha$, let $\overline{M}_i = \bigcup_{j \leq i} M_{\eta_j}$. Because of the condition on the ordering, note that for each $i < \alpha$, there is some $j \leq i$ such that $\overline{M}_i \underset{M_j}{\perp} M_{i+1}$. We inductively define an increasing elementary chain $\langle N_i : i \leq \alpha \rangle$ of elementary submodels of M such that each N_i is maximal atomic over $\overline{M}_i$ as follows:

Let $N_0 \supseteq \overline{M}_0 = M_{\langle\rangle}$ be maximal atomic. For $\beta \leq \alpha$ limit, take $N_\beta = \bigcup_{i < \beta} N_i$, which is maximal atomic over $\overline{M}_\beta$ by Lemma 5.5. Given N_i atomic over $\overline{M}_i$, note that N_i is atomic over $\overline{M}_{i+1}$ by Lemma 1.9(2), so let N_{i+1} be any maximal atomic subset of M over $\overline{M}_{i+1}$ containing $N_i \cup M_{i+1}$. That N_{i+1} is a model follows from $\aleph_0$-stability.

From the construction and Lemma 5.5, N_α is maximal atomic over $\overline{M}_\alpha = \bigcup_{\eta \in I} M_\eta$. So, it suffices to prove that $N_\alpha = M$. Suppose that this were not the case. By Fact 1.6(2), there is an element $c \in M \setminus N_\alpha$ such that $p = \mathrm{tp}(c/N_\alpha)$ is strongly regular.

Claim 1: $p \perp M_\eta$ for all $\eta \in I$.

Proof. Suppose this were not the case. Choose $\eta \in I$ $\triangleleft$-minimal such that $p \not\perp M_\eta$. Thus, either $\eta = \langle\rangle$ or $p \perp M_{\eta^-}$. By Lemma 1.7, there is an element $e \in M$ such that $\mathrm{tp}(e/M_\eta)$ is regular and non-orthogonal to p (hence orthogonal to M_{η^-} if $\eta \neq \langle\rangle$), but $e \underset{M_\eta}{\perp} N_\alpha$. This element e contradicts the maximality of C_η.

Claim 2: p is dull.

Proof. If p were eni-active, then by Lemma 5.9 we would have $p \not\perp M_\eta$ for some $\eta \in I$, contradicting Claim 1.

Finally, choose $i \leq \alpha$ least such that $p \not\perp N_i$. By Lemma 5.5, i cannot be a non-zero limit ordinal. If $i = 0$ or if $i = k + 1$, choose $q \in S(N_i)$

non-orthogonal to p and realized in M, say by e. If $i = 0$, we obtain a contradiction to the maximality of N_0 using Lemma 5.3 (with $A = M_{\langle\rangle}$). On the other hand, if $i = k+1$, then note that by the minimality of i and Claim 1, we have that $q \perp N_i$ and $q \perp M_{\eta_i}$. As well, $N_i \underset{M_j}{\perp} M_{\eta_i}$. Thus, by Lemma 5.2 we contradict the maximality of N_i.

Next, with our eye on proving Theorem 5.15, we have a series of Lemmas about dull types (i.e., regular types that are not eni-active).

Lemma 5.11 *Fix any model N, any finite $A \subseteq N$, any stationary, dull $p \in S(A)$, and any infinite A-independent $J \subseteq p(N)$. Partition J into two infinite, disjoint sets J_0, J_1, and let B be any maximal set satisfying $A \cup J_0 \subseteq B \subseteq N$ and $B \underset{A}{\perp} J$. Then B is the universe of an elementary substructure $M \subseteq B$, and N is atomic over $M \cup J_1$.*

Proof. Given N, A, p, J and B, let $M \preceq N$ be prime over B. We claim that $M \subseteq B$, so $B = M$. To see this choose $e \in M$. As $\mathrm{tp}(e/B)$ is isolated, choose a finite B_0, $A \subseteq B_0 \subseteq B$ such that $\mathrm{tp}(e/B_0) \vdash \mathrm{tp}(e/B)$. As cofinitely many $c \in J_0$ are free from B_0 over A, it follows from indiscernibility that $\mathrm{tp}(e/B_0) \vdash \mathrm{tp}(BJ_1)$. Thus, $eB \underset{A}{\perp} J_1$, so $e \in B$ follows from the maximality of B. Thus, $B = M$.

Next, let $\langle N_i : i \leq \alpha \rangle$ be a maximal continuous chain of elementary substructures of N, where N_0 is prime over $M \cup J_1$ and N_{i+1} is prime over $N_i \cup \{c_i\}$ for some $c_i \in N$ such that $\mathrm{tp}(c_i/N_i)$ is regular.

We first claim that for any $i \leq \alpha$ and any $d \in N \setminus N_i$, $\mathrm{tp}(d/N_i) \perp M$. If this were not the case, then by Fact 1.6, there would be a strongly regular $q \in S(M)$ non-orthogonal to $\mathrm{tp}(d/N_i)$. Let $q' \in S(N_i)$ be the non-forking extension of q to $S(N_i)$. By Fact 1.6 again, there would be $e \in N \setminus N_i$ realizing q'. But then, as $e \underset{M}{\perp} J_1$, e contradicts the maximality of $M = B$. Hence, any type realized in N over any N_i is orthogonal to M.

Next, we argue by induction on $i \leq \alpha$ that for every $d \in N \setminus N_i$ such that $\mathrm{tp}(d/N_i)$ is regular, is $\mathrm{tp}(d/N_i)$ is dull.

The most interesting case is when $i = 0$. As notation, enumerate $J_1 = \{a_{\langle\gamma\rangle} : \gamma < \beta\}$ and for each γ, let $M_{\langle\gamma\rangle} \preceq N_0$ be prime over $M \cup \{a_{\langle\gamma\rangle}\}$. Let I be the tree with nodes $\{\langle\rangle\} \cup \{\langle\gamma\rangle : \gamma < \beta\}$. Then $\mathfrak{d} = \langle M_\eta, a_\eta : \eta \in I \rangle$ is a regular decomposition inside N_0 and N_0 is prime over $\bigcup_{\eta \in I} M_\eta$. Now

suppose there were $d \in N \setminus N_0$ such that $q = \mathrm{tp}(d/N_0)$ were eni-active. From above, we know that $q \perp M$.

But, by Lemma 5.9, $p \perp M_{\langle\gamma\rangle}$ for some γ. This would imply that $p \not\perp M_{\langle\gamma\rangle}$, which is prime over $M \cup \{a_{\langle\gamma\rangle}\}$, yet $p \perp M$, which directly contradicts p being dull.

Now, continuing our induction, suppose that $i < \alpha$ and every element $d \in N \setminus N_i$ realizing a regular type is dull. Choose an element $e \in N \setminus N_{i+1}$ such that $q = \mathrm{tp}(e/N_{i+1})$ is regular. If q were not dull, then it would be eni-active. There are two cases: If $q \not\perp N_i$, then by Fact 1.6 there would be an eni-active $r \in S(N_i)$ that is both non-orthogonal to q and realized in N, which contradicts our inductive hypothesis. Otherwise, if $q \perp N_i$, this immediately implies that $\mathrm{tp}(c_i/N_i)$ is eni-active, which again contradicts our inductive hypothesis.

Now, by the maximality of the chain, $N_\alpha = N$. As well, it follows from the induction above that each of the types $\mathrm{tp}(c_i/N_i)$ is dull. As $\mathrm{tp}(c_i/N_i)$ is also orthogonal to M, it follows by iterating Lemma 5.4 that N is atomic over $M \cup J$.

Lemma 5.12 *Assume that N, A, p, M, J_0, J_1 are as in Lemma 5.11. Let c be a realization of the non-forking extension of p to $S(N)$ and let $N(c)$ be prime over $N \cup \{c\}$. Then $N(c)$ is atomic over $M \cup J_1 \cup \{c\}$.*

Proof. As $M \underset{A}{\perp} J_1$ and $\mathrm{tp}(c/N)$ does not fork over A, it follows that $M \underset{A}{\perp} J_1 c$. We argue that, in fact, M is a maximal subset *of* $N(c)$ with this property. Given this, the conclusion follows by quoting Lemma 5.4, with N replaced by $N(c)$ and J_1 replaced by $J_1 \cup \{c\}$.

The verification of the maximality inside $N(c)$ is an exercise in non-forking. Namely, choose any $e \in N(c)$ such that $eM \underset{A}{\perp} J_1 c$. As $J_1 \cup \{c\}$ is independent over A, we have $eMc \underset{A}{\perp} J_1$, hence $ec \underset{M}{\perp} J_1$. As N is atomic over $M \cup J_1$ by Lemma 5.4, we obtain $ec \underset{M}{\perp} N$. Combining this with the fact that $e \underset{M}{\perp} c$ yields $e \underset{M}{\perp} Nc$, hence $e \underset{N}{\perp} c$. Thus, $e \in N$ as required.

Lemma 5.13 *Suppose that $N \models T$, $\mathrm{tp}(c/N)$ is dull, and $N(c)$ is any prime model over $N \cup \{c\}$. Then N is an $L_{\infty,\aleph_0}$-elementary substructure of $N(c)$, i.e., for every finite $A \subseteq N$, $(N, a)_{a \in A} \equiv_{\infty,\aleph_0} (N(c), a)_{a \in A}$.*

Proof. Given $N \preceq N(c)$ and A, by enlarging A slightly we may assume that $p = \mathrm{tp}(c/N)$ is based and stationary on A. Choose $J = J_0 \cup J_1$ and M as in Lemma 5.11. By Lemma 5.11, N is atomic over $M \cup J_1$. As well, Lemma 5.12 imply that the hypotheses of Lemma 5.11 apply to M, $N(c)$ and $J_1 \cup \{c\}$. Thus, $N(c)$ is atomic over $M \cup J_1 \cup \{c\}$.

But now, we show that $(N, a)_{a \in M} \equiv_{\infty,\aleph_0} (N(c), a)_{a \in M}$ by exhibiting the back-and-forth system $\mathcal{F}$ consisting of all finite partial functions $f : N \to N(c)$ such that

- $f \cup id_M$ is elementary; and
- for all $e \in \mathrm{dom}(f)$, $e \in J_1$ if and only if $f(e) \in J_1 \cup \{c\}$.

The verification that $\mathcal{F}$ is a back-and-forth system is akin to the verification that any two atomic models of a complete theory are back-and-forth equivalent.

The following Corollary follows by iterating Lemma 5.13:

Corollary 5.14 *Let $\langle N_i : i \in \alpha \rangle$ be any continuous, increasing elementary chain of models of an $\aleph_0$-stable theory such that for all $i < \alpha$, N_{i+1} is prime over N_i and the realization of a dull type. Then for any $i < j < \alpha$, N_i is an $L_{\infty,\aleph_0}$-elementary substructure of N_j, i.e., for any finite $A \subseteq N_i$,*

$$(N_i, a)_{a \in A} \equiv_{\infty,\aleph_0} (N_j, a)_{a \in A}$$

Theorem 5.15 (T $\aleph_0$-stable with eni-NDOP) *Suppose $\mathfrak{d} = \langle M_\eta, a_\eta : \eta \in I \rangle$ is an eni-active decomposition of M, and choose $N \preceq M$ atomic over $\bigcup_{\eta \in I} M_\eta$. Then for any N' satisfying $N \preceq N' \preceq M$, we have that N is an $L_{\infty,\aleph_0}$-elementary substructure of N' and N' is an $L_{\infty,\aleph_0}$-substructure of M.*

Proof. Fix $\mathfrak{d}$, $N \preceq M$ atomic over $\bigcup_{\eta \in I} M_\eta$ and N' satisfying $N \preceq N' \preceq M$. As in Fact 1.6(4), choose a strongly regular resolution $\langle N_i : i \leq \alpha \rangle$, where $N_0 = N$, $N_\alpha = N'$ and, for each $i < \alpha$, N_{i+1} is prime over $N_i \cup \{c_i\}$. As well, choose a strongly regular resolution $\langle N_i : \alpha < i \leq \beta \rangle$ where $N_\alpha = N'$, $N_\beta = M$, and N_{i+1} is prime over $N_i \cup \{c_i\}$.

We argue by induction on $i \leq \beta$ that for any $d \in M \setminus N_i$ realizing a regular type in $S(N_i)$, we have $\mathrm{tp}(d/N_i)$ dull.

We begin the induction with $i = 0$. By way of contradiction, suppose that there were some $d \in M \setminus N_0$ such that $p = \mathrm{tp}(d/N_0)$ is eni-active. Since

$N_0 = N$ is atomic over $\bigcup_{\eta \in I} M_\eta$, then by Lemma 5.9 we would have $p \not\perp M_\eta$ for some $\eta \in I$. But then, by Lemma 1.7, there is an element $e \in M$ such that $\mathrm{tp}(e/M_\eta)$ is eni-active and non-orthogonal to p satisfying $e \underset{M_\eta}{\perp} N_0$. This e contradicts the maximality of C_η. Thus, any regular type in $S(N_0)$ realized in M must be dull.

It is clear from superstability that if i is a non-zero limit and our inductive hypothesis holds for all $j < i$, then it holds for i. So assume our inductive hypothesis holds for $i < \alpha$. In particular, we have $p = \mathrm{tp}(c_i/N_i)$ is dull. However, suppose there is an element $d \in M \setminus N_{i+1}$ such that $q = \mathrm{tp}(d/N_{i+1})$ is eni-active. On one hand, we cannot have $q \not\perp N_i$, lest there would be an eni-active $r \in S(N_i)$ non-orthogonal to q and realized in M. On the other hand, if the eni-active $q \perp N_i$, then this immediately implies that p is eni-active as well, which is a contradiction.

Thus, we have proved that $\mathrm{tp}(c_i/N_i)$ is dull for every $i < \beta$, and the Theorem follows from Corollary 5.14.

Corollary 5.16 *Suppose $\langle M_\eta, a_\eta : \eta \in I\rangle$ is an eni-active decomposition of both M_1 and M_2. Then $M_1 \equiv_{\infty,\aleph_0} M_2$.*

Proof. Choose any $N_1 \preceq M_1$ to be prime over $\bigcup_\eta \in IM_\eta$. By Theorem 5.15, $N_1 \equiv_{\infty,\aleph_0} M_1$. By the uniqueness of prime models, there is $N_2 \preceq M_2$ that is both isomorphic to N_1 and prime over $\bigcup_\eta \in IM_\eta$. By Theorem 5.15 again, $N_2 \equiv_{\infty,\aleph_0} M_2$ and the result follows.

The third theorem of this section involves eni decompositions of a model. Theorem 5.18 is of less interest to us, since when M^* is uncountable, each of the component submodels M_η may be uncountable as well.

Definition 5.17 A decomposition $\{M_\eta, a_\eta : \eta \in I\rangle$ inside M is *maximal atomic* if $M_{\langle\rangle}$ is a maximal atomic substructure of M and, for each $\nu \neq \langle\rangle$, M_ν is maximal atomic over $M_{\nu^-} \cup \{a_\nu\}$.

Theorem 5.18 (T $\aleph_0$-stable with eni-NDOP) *If T is $\aleph_0$-stable with eni-NDOP, then every model M is atomic over $\bigcup_{\eta\in I} M_\eta$ for every maximal atomic, eni decomposition $\{M_\eta : \eta \in I\}$ of M.*

Proof. We argue as in the proof of Theorem 5.10, producing a continuous, elementary sequence $\langle N_i : i \leq \alpha\rangle$, where each N_i is maximal atomic

over $\overline{M}_i$. Note that here, however, $N_0 = M_{\langle\rangle}$ by the maximality of $M_{\langle\rangle}$. We know that N_α is atomic over $\bigcup_{\eta \in I} M_\eta$, so it suffices to prove $N_\alpha = M$. If not, choose a strongly regular $p \in S(N_\alpha)$ realized in M.

We argue by cases. First, if p were eni, then (as it is also eni-active) we would have $p \not\perp M_\eta$ for some $\eta \in I$ by Lemma 5.9. But then, by Lemma 1.7 we would have a realization $e \in M$ such that $\mathrm{tp}(e/M_\eta)$ is eni, but $e \underset{M_\eta}{\perp} N_\alpha$, which contradicts the maximality of C_η. Thus, p is not eni.

Now, arguing as in the proof of Theorem 5.10, choose $i \leq \alpha$ minimal such that $p \not\perp N_i$. By superstability, i is either zero or $i = k+1$ for some k. As before, we obtain a contradiction, either by Lemma 5.3 (when $i = 0$) or by Lemma 5.2 when $i = k+1$.

We close this section with an application of Theorem 5.10. The main point of the proof of Corollary 5.19 is that models that are atomic over an independent tree of countable models have a large number of partial automorphisms.

Corollary 5.19 *If T is $\aleph_0$-stable and eni-NDOP, then T cannot have OTOP.*

Proof. By way of contradiction suppose that there were sufficiently large cardinal κ and a model M^* containing a sequence $\langle (b_\alpha, c_\alpha) : \alpha < \kappa \rangle$ and a type $p(x, y, z)$ such that for all $\alpha, \beta < \kappa$,

$$M^* \text{ realizes } p(x, b_\alpha, c_\beta) \quad \text{if and only if} \quad \alpha < \beta$$

For each pair $\alpha < \beta$, fix a realization $a_{\alpha,\beta}$ of $p(x, b_\alpha, c_\beta)$. Choose a prime, regular decomposition $\langle M_\eta, a_\eta : \eta \in I \rangle$ of M^*. Note that each of the models M_η is countable. By Theorem 5.10, M^* is atomic over $\bigcup_{\eta \in I} M_\eta$, so for each pair $\alpha < \beta$ we can choose a finite $e_{\alpha,\beta}$ from $\bigcup_{\eta \in I} M_\eta$ such that $\mathrm{tp}(a_{\alpha,\beta}/b_\alpha, c_\beta \bigcup_{\eta \in I} M_\eta)$ is isolated by a formula $\theta(x, b_\alpha, c_\beta, e_{\alpha,\beta})$. We will eventually find a pair $\alpha < \beta$ and e^* from $\bigcup_{\eta \in I} M_\eta$ such that

$$\mathrm{tp}(b_\beta, c_\alpha, e^*) = \mathrm{tp}(b_\alpha, c_\beta, e_{\alpha,\beta})$$

This immediately leads to a contradiction, as $\theta(x, b_\beta, c_\alpha, e^*)$ would be realized in M^* and any realization of it also realizes $p(x, b_\beta, c_\alpha)$, contrary to our initial assumptions.

We will obtain these $\alpha < \beta$ and e^* by successively passing from our sequence to sufficiently long subsequences, each time adding some amount

of homogeneity. First, for each α, choose a finite subtree $J_\alpha \subseteq I$ such that $\mathrm{tp}(b_\alpha c_\alpha / \bigcup_{\eta \in I} M_\eta)$ does not fork and is as stationary as possible over J_α. By an argument akin to the Δ-system lemma, by passing to a subsequence we may assume that there is an $\eta^* \in I$ such that $J_\alpha \cap J_\beta = \{\nu : \nu \trianglelefteq \eta\}$ for all $\alpha \neq \beta$. For each α, let M_α^J be the countable set $\bigcup_{\gamma \in J_\alpha} M_\gamma$. As well, let ν_α be the (unique) immediate successor of η^* contained in J_α, let $H_\alpha = \{\gamma \in I : \nu_\alpha \trianglelefteq \gamma\}$, and let $M_\alpha = \bigcup_{\gamma \in H_\alpha} M_\gamma$. Note that the sets H_α are pairwise disjoint and the independence of the tree implies that the sets $\{M_\alpha : \alpha \in \kappa\}$ are independent over M_{η^*}. By trimming further, we may additionally assume that each of the J_α's are tree isomorphic over η^*, and that the sets M_α are isomorphic over M_{η^*}.

Next, for each $\alpha < \beta$, partition each sequence $e_{\alpha,\beta}$ into three subsequences $r_{\alpha,\beta} \subseteq M_\alpha$, $s_{\alpha,\beta} \subseteq M_\beta$, and $t_{\alpha,\beta}$ disjoint from $M_\alpha \cup M_\beta$.

By the Erdös-Rado Theorem, we can pass to a subsequence such that for all $\alpha < \beta < \gamma$ we have:

- The partitions coincide, i.e., for each i, the i^{th} coordinate of $e_{\alpha,\beta} \in r_{\alpha,\beta}$ iff the i^{th} coordinate of $e_{\beta,\gamma} \in r_{\beta,\gamma}$;
- $\mathrm{tp}(t_{\alpha,\beta}/M_{\eta^*})$ is constant;
- $\mathrm{tp}(r_{\alpha,\beta}/M_\alpha^J)$ is constant; and
- $\mathrm{tp}(s_{\alpha,\beta}/M_\beta^J)$ is constant.

Additionally, by trimming the sequence still further, we may insist that for all pairs $\alpha < \beta$, there is $r^* \in H_\beta$ such that $\mathrm{tp}(r_{\alpha,\beta} M_\alpha^J / M_{\eta^*}) = \mathrm{tp}(r^* M_\beta^J / M_{\eta^*})$ and there is $s^* \in H_\alpha$ such that $\mathrm{tp}(s_{\alpha,\beta} M_\beta^J / M_{\eta^*}) = \mathrm{tp}(s^* M_\alpha^J / M_{\eta^*})$.

Finally, fix any such $\alpha < \beta$. By independence, we have

$$\mathrm{tp}(M_\alpha^J, M_\beta^J, r_{\alpha,\beta}, s_{\alpha,\beta}, t_{\alpha,\beta}) = \mathrm{tp}(M_\beta^J, M_\alpha^J, r^*, s^*, t_{\alpha,\beta})$$

Let e^* be the sequence formed from $r^* s^* t_{\alpha,\beta}$. As each of b_α and c_β are dominated by M_α^J and M_β^J, respectively over M_{η^*}, it follows that $b_\alpha c_\beta \underset{M_\alpha^J M_\beta^J}{\perp} \bigcup_{\eta \in I} M_\eta$, so $\mathrm{tp}(b_\alpha, c_\beta, e_{\alpha,\beta}) = \mathrm{tp}(b_\beta, c_\alpha, e^*)$, completing the proof.

6 Borel completeness of eni-NDOP, eni-deep theories

Throughout this section, we assume that T is $\aleph_0$-stable with eni-NDOP, and is eni-deep.

All decompositions we consider in this section are eni-active

Fix an eni-active chain $\langle M_i, a_i : i \in \omega \rangle$ of length ω that witnesses the infinite depth. As $\mathrm{tp}(a_{i+1}/M_i)$ is eni-active, for every i, there is an integer $k = k(i) > i$ and an eni-active chain $\mathcal{C}_k = \langle N^k_j, b^k_j : j \leq k \rangle$ and an ENI $q_k \in S(N^k_k)$ satisfying $q^k \perp N^k_{k-1}$ and for every $j \leq i$, $N^k_j = M_j$ and $b^k_j = a_j$. (We say that $\mathcal{C}_k$ *extends* i if this last clause holds.)

We will use this data to code arbitrary subtrees of $\mathcal{T} \subseteq \lambda^{<\omega}$ into models $M(\mathcal{T})$ preserving isomorphism in both directions. The 'reverse direction' i.e., showing that $M(\mathcal{T}_1) \cong M(\mathcal{T}_2)$ implying $(\mathcal{T}_1, \trianglelefteq) \cong (\mathcal{T}_2, \trianglelefteq)$ is quite involved and uses a 'black box' in the form of Theorem 6.19 of [10]. We begin by recalling a number of definitions that appear there. As we are concerned with eni-active decompositions, we should take $\mathbf{P}$ to be the set of eni-active, regular types. Note that $\mathbf{P}^r = \mathbf{P}$ in the notation of [10].

Definition 6.1 Given a tree $I \subseteq Ord^{<\omega}$, a *large subtree* of I is a non-empty subtree $J \subseteq I$ such that for each $\eta \in J$, $Succ_I(\eta) \setminus J$ is finite. We say that two trees I_1 and I_2 are *almost isomorphic* if there exist large subtrees $J_1 \subseteq I_1$ and $J_2 \subseteq I_2$ such that $(J_1, \trianglelefteq) \cong (J_2, \trianglelefteq)$.

A tree I has *infinite branching* if, for every $\eta \in I$, $Succ(\eta)$ is either infinite or empty. If a tree I has infinite branching, for any integer k, we say a node $\eta \in I$ has *uniform depth* k if, for every maximal branch of $\{\nu \in I : \eta \trianglelefteq \nu\}$ has length exactly k. A node η *often has unbounded depth* if, for every large subtree $J \subseteq I$ with $\eta \in J$, there is an infinite branch in J containing η.

A node η is an (m,n)*-cusp* if there are infinite sets $A_m, A_n, B \subseteq Succ(\eta)$ such that

1. the set $A_m \cup A_n$ is pairwise E_η-equivalent;
2. each $\delta \in A_m$ has uniform depth m;
3. each $\rho \in A_n$ has uniform depth n; and

4. each $\gamma \in B$ is often unbounded.

A *cusp* is an (m, n)-cusp for some $m \neq n$.

Definition 6.2 Suppose $S \subseteq \mathbf{P}$ and $\mathfrak{d} = \langle M_\eta, a_\eta : \eta \in I\rangle$ is a $\mathbf{P}$-decomposition. We say $\mathfrak{d}$ *supports* S if, for every $q \in S$ there is $\eta(q) \in \max(I) \setminus \{\langle\rangle\}$ such that $q \not\perp M_{\eta(q)}$, but $q \perp M_{\eta(q)^-}$. If $\mathfrak{d}$ supports S, then we let $\mathrm{Field}(S) := \{\eta(q) : q \in S\}$ and $I^S := \{\nu \triangleleft \eta : \eta \in \mathrm{Field}(S)\}$.

Definition 6.3 Suppose $S \subseteq \mathbf{P}$, a model M, and fix a function $\Phi : \omega \to \omega$. We say that an eni-active decomposition $\mathfrak{d} = \langle M_\eta, a_\eta : \eta \in I\rangle$ of M is $\mathbf{P}$-*finitely saturated* if, for every finite $A \subseteq M$ and $p \in S(A) \cap \mathbf{P}$, there is $\eta \in I$ such that $p \not\perp M_\eta$.

The decomposition $\mathfrak{d}$ is (S, Φ)-*simple* if

1. $\mathfrak{d}$ is $\mathbf{P}$-finitely saturated;
2. $\mathfrak{d}$ supports M (hence I^S is defined);
3. for all $\eta, \nu \in I^S$
   (a) if $\eta^- = \nu^- = \mu$, then $\mathrm{tp}(a_\eta/M_\mu) = \mathrm{tp}(a_\nu/M_\mu)$;
   (b) $Succ_{I^S}(\eta)$ is empty or infinite (hence I^S has infinite branching);
   (c) η is either of some finite uniform depth or is a cusp;
   (d) if η is an (m, n)-cusp, then $\Phi(m - n) = \lg(\eta)$.

Theorem 6.19 from [10], which we take as a black box, states:

Theorem 6.4 *Suppose $S \subseteq \mathbf{P}$, a model M, and a function $\Phi : \omega \to \omega$ are given. If $\mathfrak{d}_1$ and $\mathfrak{d}_2$ are both (S, Φ)-simple decompositions of M, then the trees I_1^S and I_2^S are almost isomorphic.*

With our eye on applying Theorem 6.4, we massage the data we were given at the top of this section.

Let $\mathcal{U} = \{k \in \omega : k = k(i) \text{ for some } i\}$. As $\mathcal{U}$ is infinite, by passing to an infinite subset, we may additionally assume that if $n < m$ are from $\mathcal{U}$, then $m > 2n$. It follows from this that for all pairs $n < m$, $n' < m'$ from $\mathcal{U}$,

$$m - n = m' - n' \qquad \text{if and only if} \qquad m = m' \text{ and} n = n'$$

Next, it is routine to partition $\mathcal{U}$ into infinitely many infinite sets V_i for which $k > i$ for every $k \in V_i$.

Fix an integer i. An 'i-tree' is a subtree of $\omega^{<\omega}$ with a unique 'stem' $\{\langle 0^j \rangle : j < i\}$ of length i. As an example, for each $k \in V_i$, let

$$I_i(k) := \{\eta \in \omega^{\leq k} : \text{ for all } j < i, \text{ if } \lg(\eta) > j, \text{ then } \eta(j) = 0\}$$

If I and J are both i-trees (say with disjoint universes) the *free join of I and J over i, $I \oplus_i J$,* is the i-tree with universe $(I \cup J)/\sim$, where for each $j < i$, the (unique) nodes of I and J of length j are identified, and every other $\sim$-class is a singleton. To set notation, for $n < m$ from V_i, let $I_i(n,m) := I_i(n) \oplus_i I_i(m)$. We associate an eni-active decomposition

$$\mathfrak{d}(n,m) := \langle N_\eta, b_\eta : \eta \in I_i(n,m)\rangle$$

satisfying:

- for $\lg(\eta) < i$, $N_\eta = M_i$ and $b_\eta = a_i$;
- if $k(\eta) = n$ when $\eta \in I_i(n)$ and $k(\eta) = m$ when $\eta \in I_i(m)$, then $N_\eta \cong N^{k(\eta)}_{\lg(\eta)}$ and $\operatorname{tp}(b_\nu/N_{\nu^-}) = \operatorname{tp}(b^{k(\nu)}_{\lg(\nu)}/N^{k(\nu)}_{\lg(\nu^-)})$.

In particular, as $\mathfrak{d}(n,m)$ is a decomposition, $\{N_\eta : \eta \in I_i(n,m)\}$ form an independent tree of models.

Still with i fixed, choose disjoint, 4-element sets $\{n(\delta^+), m(\delta^+), n(\delta^-), m(\delta^-)\}$ from V_i for each $\delta \in \omega^i$ such that $n(\delta+) < m(\delta^+)$ and $n(\delta^-) < m(\delta^-)$.

Now, for each $\delta \in \omega^{<\omega}$, let $\operatorname{diff}(\delta^+) = m(\delta^+) - n(\delta^+)$ and $\operatorname{diff}(\delta^-) = m(\delta^-) - n(\delta^-)$. It follows from our thinness conditions on $\mathcal{U}$ (and the disjointness of the sets V_i) that the set $D = \{\operatorname{diff}(\delta^+), \operatorname{diff}(\delta^-) : \delta \in \omega^{<\omega}\}$ is without repetition. Let $\Phi : \omega \to \omega$ be any function such that for every $\delta \in \omega^{<\omega}$,

$$\Phi(\operatorname{diff}(\delta^+)) = \Phi(\operatorname{diff}(\delta^-)) = \lg(\delta)$$

To ease notation, for each $\delta \in \omega^{<\omega}$, let $I(\delta^+) = I_i(n(\delta^+), m(\delta^+))$ and $\mathfrak{d}(\delta^+) = \mathfrak{d}(n(\delta^+), m(\delta^+))$, with analogous definitions for $I(\delta^-)$ and $\mathfrak{d}(\delta^-)$.

Next, let $I_0 := (\lambda \times \omega)^{<\omega}$. We denote elements of I_0 by pairs (η, δ). Note that $\lg(\eta) = \lg(\delta)$ for all $(\eta, \delta) \in I_0$. Let $\mathfrak{d}_0$ denote the eni-active decomposition $\langle M_{(\eta,\delta)}, a_{(\eta,\delta)} : (\eta, \delta) \in I_0\rangle$, where $M_{(\eta,\delta)} \cong M_{\lg(\eta)}$ via a map $f_{(\eta,\delta)}$, and $f_{(\eta,\delta)}(a_{(\eta,\delta)}) = a_{\lg(\eta)}$.

With all of the above as a preamble, we are now ready to code subtrees of $\lambda^{<\omega}$ into models of our theory.

Theorem 6.5 (T $\aleph_0$-stable, eni-NDOP, eni-deep.) *For any $\lambda \geq \aleph_0$, there is a λ-Borel embedding $\mathcal{T} \mapsto M(\mathcal{T})$ of subtrees of $\lambda^{<\omega}$ into models of size λ satisfying*

$$(\mathcal{T}_1, \trianglelefteq) \cong (\mathcal{T}_2, \trianglelefteq) \qquad \textit{if and only if} \qquad M(\mathcal{T}_1) \cong M(\mathcal{T}_2)$$

Proof. Fix a cardinal $\lambda \geq \aleph_0$. We describe the map $\mathcal{T} \mapsto M(\mathcal{T})$. Fix a subtree $\mathcal{T} \subseteq \lambda^{<\omega}$. Begin by letting $\delta_0(\mathcal{T})$ be the eni-active decomposition formed by beginning with the decomposition $\mathfrak{d}_0$ and simultaneously adjoining a copy of $\mathfrak{d}(\delta^+)$ to every node $(\eta, \delta) \in I_0$ for which $\eta \in \mathcal{T}$, as well as adjoining a copy of $\mathfrak{d}(\delta^-)$ to every node $(\eta, \delta) \in I_0$ for which $\eta \notin \mathcal{T}$. Let $I_0(\mathcal{T})$ denote the index tree of $\mathfrak{d}_0(\mathcal{T})$. Let $M_0(\mathcal{T})$ be prime over $\bigcup\{N_\nu : \nu \in I_0(\mathcal{T})\}$. For each $\nu \in \max(I_0(\mathcal{T}))$, let $q_\nu \in S(N_\nu)$ be the ENI-type conjugate to $q_{\lg(\nu)} \in S(N^{\lg(\nu)}_{\lg(\nu)})$ and let $S = \{q_\nu : \nu \in \max(I_0(\mathcal{T}))\}$. Because of the independence of the tree and the fact that $M_0(\mathcal{T})$ is prime over the tree, each q_ν has finite dimension in $M_0(\mathcal{T})$.

Next, we recursively construct an elementary chain $\langle M_n(\mathcal{T}) : n \in \omega\rangle$ and a sequence $\langle \mathfrak{d}_n(\mathcal{T}) : n \in \omega\rangle$ as follows. We have already defined $M_0(\mathcal{T})$ and $\mathfrak{d}_0(\mathcal{T})$, so assume $M_n(\mathcal{T})$ is defined and $\mathfrak{d}_n(\mathcal{T})$ is an eni-active decomposition of $M_n(\mathcal{T})$ extending $\mathfrak{d}_0(\mathcal{T})$. Let R_n consist of all $p \in S(M_n(\mathcal{T})) \cap \mathbf{P}$ satisfying $p \perp S$. Let $J_n := \{a_p : p \in R_n\}$ be a $M_n(\mathcal{T})$-independent set of realizations of each $p \in R_n$. For each $p \in R_n$, there is a $\triangleleft$-minimal $\eta(p) \in I_n(\mathcal{T})$ such that $p \not\perp N_{\eta(p)})$. Let N_p be prime over $N_{\eta(p)} \cup \{a_p\}$. Let $\mathfrak{d}_{n+1}(\mathcal{T})$ be the natural extension of $\mathfrak{d}_n(\mathcal{T})$ formed by affixing each N_p as an immediate successor of $N_{\eta(p)}$, and let $M_{n+1}(\mathcal{T})$ be prime over the independent tree of models in $\mathfrak{d}_{n+1}(\mathcal{T})$.

Finally, let $\mathfrak{d}(\mathcal{T}) := \bigcup_{n\in\omega} \mathfrak{d}_n(\mathcal{T})$ and let $M(\mathcal{T})$ be prime over $\mathfrak{d}(\mathcal{T})$. As notation, let $I(\mathcal{T})$ denote the index tree of $\mathfrak{d}(\mathcal{T})$.

The following facts are easily established:

1. A type $p \in S(M(\mathcal{T})) \cap \mathbf{P}$ has finite dimension in $M(\mathcal{T})$ if and only if $p \not\perp S$;

2. $\mathfrak{d}(\mathcal{T})$ is $\mathbf{P}$-finitely saturated;

3. $\mathfrak{d}(\mathcal{T})$ supports S and $I^S(\mathcal{T}) = I_0(\mathcal{T})$;

4. $I^S(\mathcal{T})$ is infinitely branching; and

5. for $\nu \in I^S(\mathcal{T})$,

   - ν is a cusp if and only if $\nu \in I_0$. In particular, if $\nu = (\eta, \delta)$ and $\eta \in \mathcal{T}$, then ν is an $(m(\delta^+), n(\delta^+))$-cusp, and $\eta \notin \mathcal{T}$, then ν is an $(m(\delta^-), n(\delta^-))$-cusp;
   - if $\nu \in I_0(\mathcal{T}) \setminus I_0$, then ν is of uniform finite depth.

In particular, $\mathfrak{d}(\mathcal{T})$ is an (S, Φ)-simple decomposition of $M(\mathcal{T})$.

Main Claim: If $M(\mathcal{T}_1) \cong M(\mathcal{T}_2)$, then $(\mathcal{T}_1, \trianglelefteq) \cong (\mathcal{T}_2, \trianglelefteq)$.

Proof. Suppose that $f : M(\mathcal{T}_1) \to M(\mathcal{T}_2)$ is an isomorphism. Then the image of $\mathfrak{d}(\mathcal{T}_1)$ under f is a decomposition of $M(\mathcal{T}_2)$ with index tree $I(\mathcal{T}_1)$. As well, $\mathfrak{d}_(\mathcal{T}_2)$ is also a decomposition of $M(\mathcal{T}_2)$ with index tree $I(\mathcal{T}_2)$. If, for $\ell = 1, 2$, we let S_ℓ denote the non-orthogonality classes of ENI types of finite dimension in $M(\mathcal{T}_\ell)$, then as isomorphisms preserve types of finite dimension, $f(S_1) = S_2$ setwise. It follows that both $f(\mathfrak{d}_1)$ and $\mathfrak{d}_2$ are both $(S_2.\Phi)$-simple decompositions of $M(\mathcal{T}_2)$. Thus, by Theorem 6.4, the trees $I_0(\mathcal{T}_1)$ and $I_0(\mathcal{T}_2)$ are almost isomorphic.

Fix large subtrees $J_\ell \subseteq I_0(\mathcal{T}_\ell)$ and a tree isomorphism $h : J_1 \to J_2$. Note that for $\ell = 1, 2$, a node $\nu \in J_\ell$ has uniform depth k in J_ℓ if and only if ν has uniform depth k in $I_0(\mathcal{T}_\ell)$. It follows that h maps cusps to cusps, and more precisely, (m, n)-cusps to (m, n)-cusps. Thus, the restriction h' of h to $J_1 \cap (\lambda \times \omega)^{<\omega}$ is a tree isomorphism mapping onto $J_2 \cap (\lambda \times \omega)^{<\omega}$ that sends (m, n)-cusps to (m, n)-cusps. However, as the pairs (m, n) uniquely identify $\delta \in \omega^{<\omega}$ and even δ^+ and δ^-, it follows that $h'(\eta, \delta) = (\eta^*, \delta)$ for every $(\eta, \delta) \in \mathrm{dom}(h')$. As well, if we let

$$P_\ell := \{(\eta, \delta) \in J_\ell \cap (\lambda \times \omega)^{<\omega} : (\eta, \delta) \text{ is a } \delta^+\text{-cusp}\}$$

then h' maps P_1 onto P_2 as well. Recalling that from our construction, $(\eta, \delta) \in P_\ell$ if and only if $\eta \in \mathcal{T}_\ell$, we have that for every $(\eta, \delta) \in \mathrm{dom}(h')$

$$\text{if } h'(\eta, \delta) = (\eta^*, \delta), \text{ then } \eta \in \mathcal{T}_1 \text{ if and only if } \eta^* \in \mathcal{T}_2.$$

To finish, we recursively construct maps $h^* : \lambda^{<\omega} \to \lambda^{<\omega}$ and $\delta^* : \lambda^{<\omega} \to \omega^{<\omega}$ satisfying:

1. $(\eta, \delta^*(\eta)) \in J_1$;

2. $h^*(\eta) = \eta^*$ if and only if $h'(\eta, \delta^*(\eta)) = (\eta^*, \delta^*(\eta))$;

3. for all η and all $\alpha, \alpha' \in \lambda$, $\delta^*(\eta\hat{\ }\langle\alpha\rangle) = \delta^*(\eta\hat{\ }\langle\alpha'\rangle)$; and

4. for all $\eta \in \lambda^{<\omega}$, $\alpha, \beta \in \lambda$, $(\eta\hat{\ }\langle\alpha\rangle, \delta^*(\eta\hat{\ }\langle\alpha\rangle)) \in J_1$ and $(h^*(\eta)\hat{\ }\langle\beta\rangle, \delta^*(h^*(\eta)\hat{\ }\langle\beta\rangle)) \in J_2$.

To accomplish this, first let $\delta^*(\langle\rangle) = \langle\rangle$. Given that $\delta^*(\eta)$ is defined, the definition of $h^*(\eta)$ is given by Clause (2). As $(\eta, \delta^*(\eta) \in J_1$ and since J_ℓ are large subtrees of $I_0(\mathcal{T}_\ell)$, it follows that there is $\delta' \in Succ(\delta^*(\eta))$ such that Clauses (3) and (4) hold for all $\alpha, \beta \in \lambda$. Define $\delta^*(\eta\hat{\ }\langle\alpha\rangle) = \delta'$ for every α and define $h^*(\eta\hat{\ }\langle\alpha\rangle)$ according to Clause (2).

It is easily checked that $h^* : \lambda^{<\omega} \to \lambda^{<\omega}$ is a tree isomorphism. Additionally, as h' mapped P_1 onto P_2, it follows that the restriction of h^* to $\mathcal{T}_1$ is a tree isomorphism between $(\mathcal{T}_1, \trianglelefteq)$ and $(\mathcal{T}_2, \trianglelefteq)$.

Corollary 6.6 *If T is $\aleph_0$-stable with eni-NDOP and is eni-deep, then T is Borel complete. Moreover, for every infinite cardinal λ, T is λ-Borel complete for $\equiv_{\infty,\aleph_0}$.*

Proof. If T has eni-DOP, then this is literally Corollary 4.13. If T has eni-NDOP, then the proof is exactly like the proof of Corollary 4.13, using Theorem 6.5 in place of Theorem 4.12.

7 Main gap for models of $\aleph_0$-stable theories modulo $\equiv_{\infty,\aleph_0}$

In this brief section, we combine our previous results to exhibit a dichotomy among $\aleph_0$-stable theories.

Definition 7.1 For T any theory and λ an infinite cardinal, let $Mod_\lambda(T)$ denote the set of models of T with universe λ.

- For T any theory and λ any cardinal, $I_{\infty,\aleph_0}(T, \lambda)$ denotes the maximum cardinality of any pairwise non-$\equiv_{\infty,\aleph_0}$ collection from $Mod_\lambda(T)$.

- For any $M \models T$ of size λ, the *Scott height of* M, $SH(M)$ is the least ordinal $\alpha < \kappa^+$ such that for any model N, $N \equiv_\alpha M$ implies $N \equiv_{\alpha+1} M$.

Theorem 7.2 *The following conditions are equivalent for any $\aleph_0$-stable theory T:*

1. *For all infinite cardinals λ, $I_{\infty,\aleph_0}(T,\lambda) = 2^\lambda$;*
2. *For all infinite cardinals λ, $\sup\{SH(M) : M \in Mod_\lambda(T)\} = \lambda^+$;*
3. *T either has eni-DOP or is eni-deep.*

Proof. The equivalence of (1) $\Leftrightarrow$ (2) is the content of [7].

(3) $\Rightarrow$ (1) : Fix any infinite cardinal λ. If T has either of these properties, then by Corollary 4.13 or Corollary 6.6, T is λ-Borel complete. However, it is well known (see e.g., [8]) that there is a family of 2^λ pairwise non-$\equiv_{\infty,\aleph_0}$ directed graphs with universe λ. It follows immediately that $I_{\infty,\aleph_0}(T,\lambda) = 2^\lambda$ in either case.

(1) $\Rightarrow$ (3) : Assume that T is $\aleph_0$-stable, with eni-NDOP and eni-shallow (i.e., not eni-deep). Then, by Corollary 5.16, models of T are determined by up to $\equiv_{\infty,\aleph_0}$-equivalence by their prime, eni-active decompositions. Thus, it suffices to count the number of prime, eni-active decompositions up to isomorphism.[1]

To obtain this count, first note that if T is eni-shallow, then as in Theorem X 4.4 of [6] (which builds on VII, Section 5 of [6]), the depth of any index tree of an eni-active decomposition is an ordinal $\beta < \omega_1$. In any prime decomposition, each of the models M_η is countable, hence there are at most $2^{\aleph_0}$ isomorphism types. So, as a weak upper bound, if $\lambda = \aleph_\alpha$, then the number of prime, eni-active decompositions of depth β of a model of size λ is bounded by $\beth_{(|\alpha|+|\beta|)^+}$. [Similar counting arguments appear in Theorem X 4.7 of [6].] From this, we conclude that for some cardinals λ, $I_{\infty,\aleph_0}(T,\lambda) < 2^\lambda$.

[1] We say that two eni-active decompositions $\mathfrak{d}_1 = \langle M^1_\eta, a^1_\eta : \eta \in I_1\rangle$ and $\mathfrak{d}_2 = \langle M^2_\eta, a^2_\eta : \eta \in I_2\rangle$ are isomorphic if there is a tree isomorphism $f : (I_1, \trianglelefteq) \cong (I_2, \trianglelefteq)$ and an elementary bijection $f^* : \bigcup_{\eta\in I_1} M^1_\eta \to \bigcup_{\eta \in I_2} M^2_\eta$ such that, for each $\eta \in I_1$, $f^*|_{M^1_\eta}$ maps M^1_η isomorphically onto $M^2_{f(\eta)}$.

A Packing problems for bipartite graphs

A bipartite graph A consists of a set of vertices, which are partitioned into two sets $L(A)$ and $R(A)$, together with a binary, irreflexive edge relation $E(A) \subseteq L(A) \times R(A)$. We say that A is *complete bipartite* if the set of edges $E(A) = L(A) \times R(A)$. We call A *balanced* if $||L(A)| - |R(A)|| \leq 1$.

Define a function $e^* : \omega \to \omega$ by $e^*(2b) = b^2$ and $e^*(2b+1) = b(b+1)$ for all $b \in \omega$. A classical packing problem asserts:

Fact A.1 *A bipartite graph A with at most $c \geq 2$ vertices has at most $e^*(c)$ edges, with equality holding if and only if $|A| = c$ and A is complete and balanced.*

For a bipartite graph A, let $v(A)$, $e(A)$, and $CC(A)$ denote the number of vertices, edges, and connected components of A, respectively. Recall that in the discussion prior to the statement of Proposition 4.4, we defined an $m_1 \times m_2$ bipartite graph A to be almost ℓ-complete if $|m_i - \ell| \leq 0.01\ell$ for each $i = 1, 2$ and each vertex has valence at least 0.9ℓ.

Fact A.2 *Suppose that N is a given integer and $\ell >> N$ (explicit bounds on ℓ in terms of N can be found from the proof). If A is any bipartite graph with $v(A) \leq 2\ell + N$ and $e(A) \geq \ell^2 - N$, then A is almost ℓ-complete.*

Proof. The new statistic we investigate in this Appendix is $k(A)$, which we define to be $v(A) - CC(A)$. Two special cases are that $v(A) = k(A) + 1$ for any connected bipartite graph A, and that any null bipartite graph B has $k(B) = 0$.

We wish to find an analogue of Fact A.2 in which the upper bound on $v(A)$ is replaced by an upper bound on $k(A)$.

If A and B are each bipartite graphs with disjoint sets of vertices, then $A \coprod B$ denotes their *disjoint union*. It is the bipartite graph C whose vertices are the union of the vertices of A and B, and $E(C) = E(A) \cup E(B)$.

Note that all of our statistics are additive with respect to disjoint unions. For example, for $x \in \{n, e, CC, k\}$, $x(A \coprod B) = x(A) + x(B)$. Thus, if A is any bipartite graph and B is null, then $k(A \coprod B) = k(A)$. The proof of the following Lemma is routine.

Lemma A.3 *Suppose A and B are disjoint, and are each complete, balanced, bipartite graphs with $k(A) \geq k(B) \geq 1$. Let A^+ and B^- be disjoint, complete, balanced bipartite graphs with $k(A^+) = k(A)+1$ and $k(B^-) = k(B)-1$. Then $k(A^+ \coprod B^-) = k(A \coprod B)$ and $e(A^+ \coprod B^-) \geq e(A \coprod B)$.*

One Corollary follows immediately by combining Fact A.1 with Lemma A.3.

Corollary A.4 *For all positive integers a and all bipartite graphs A with $k(A) \leq a$, $e(A) \leq e^*(a+1)$, with equality holding if and only if $A = B \coprod C$, wit B complete and balanced, and C null (and may be empty).*

Next, given a pair of integers c, d, let $f(c, d)$ be the least integer such that $e(A) \leq f(c, d)$ for all bipartite graphs of the form $A = B \coprod C$, where $k(B) \leq c$ and $k(C) \leq d$.

Lemma A.5

1. *For all $c, d \in \omega$, $f(c, d) = e^*(c+1) + e^*(d+1)$; and*
2. *If $1 \leq d \leq c$, then $f(c+1, d-1) \geq f(c, d)$.*

Proof. The first statement follows by applying Fact A.1 to each of B and C, while the second follows from Lemma A.3.

Proposition A.6 *If $\ell > W^2/4$ and A is a bipartite graph satisfying $k(A) \leq 2\ell + W$ and $e(A) \geq \ell^2$, then A contains a connected subgraph $B \subseteq A$ with at least $\ell^2 - W^2/4$ edges and at most $2\ell + W$ vertices.*

Proof. Let Φ be the set of all A such that $k(A) \leq 2\ell + W$ and A does not have any connected component B with $k(B) \geq 2\ell - 1$. Among all such A, choose $A^* \in \Phi$ so as to maximize the number of edges $e(A^*)$. By Lemma A.5,

$$e(A^*) \leq f(2\ell - 1, W+1) = e^*(2\ell - 1) + e^*(W+1) \leq \ell(\ell - 1) + W^2/4 < \ell^2$$

Thus, our given graph $A \notin \Phi$, so A has a connected component B with $k(B) \geq 2\ell - 1$. Now, if we decompose A as $A = B \coprod C$, then $k(C) \leq W + 1$, so by Lemma A.3, $e(C) \leq e^*(W+1) \leq W^2/4$.

Since $\ell^2 \leq e(A) = e(B) + e(C)$, this implies $e(B) \geq \ell^2 - W^2/4$. But, since B is connected, $v(B) = k(B) + 1$ and $k(B) \leq k(A) \leq 2\ell + W$, so B has at most $2\ell + W$ vertices.

The following Corollary follows immediately by combining Proposition A.6 with Fact A.2.

Corollary A.7 *If $\ell >> W$ and A is a bipartite graph satisfying $k(A) \leq 2\ell + W$ and $e(A) \geq \ell^2$, then A has an almost ℓ-complete subgraph.*

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